DIRECT LIMIT TOPOLOGIES AND A TOPOLOGICAL CHARACTERIZATION OF LF-SPACES

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1. Topological versus uniform direct limits.

By the direct limit $\text{t-}\varinjlim X_n$ of a tower

$$X_0 \subset X_1 \subset X_2 \subset \cdots$$

of topological spaces we understand the union $X = \bigcup_{n\in\omega} X_n$ endowed with the largest topology turning the identity inclusions $X_n \to X$ into continuous maps. Such a tower (X_n) is called *closed* if each space X_n is closed in X_{n+1}.

By analogy we can define the direct limit $\text{u-}\varinjlim X_n$ of a tower (X_n) of uniform spaces: this is the union $X = \bigcup_{n\in\omega} X_n$ endowed with the uniformity turning the identity inclusions $X_n \to X$ into uniformly continuous maps.

The topology of $\text{t-}\varinjlim X_n$ can be easily described: it consists of all subsets $U \subset X$ whose intersection $U \cap X_n$ are open in X_n for all $n \in \omega$.

The describtion of the topology of the uniform direct limit is less trivial and relies on the following operation over relations. Given a set X and two relations $U, V \subset X^2$ let

$$U + V = \{(x, z) \in X^2 : \exists y \in X \ \ (x, y) \in U,\ (y, z) \in V\}.$$

By induction this operation can be extended over sequences $(U_n)_{n\in\omega}$ of subsets of X^2 as follows:

$$\sum_{i\le n} U_i = U_0 + \cdots + U_n \text{ and } \sum_{i\in\omega} U_i = \bigcup_{n\in\omega}\sum_{i\le n} U_i.$$

For a point $a \in X$, a subset $A \subset X$, and a relation $U \subset X^2$ let

$$B(a, U) = \{x \in X : (x, a) \in U\} \text{ and } B(A, U) = \bigcup_{a\in A} B(a, U)$$

denote the U-balls around a and A, respectively.

For a uniform space X by $\mathcal{U}_X$ we denote the uniformity of X.

Given a tower

$$X_0 \subset X_1 \subset X_2 \subset \ldots$$

of sets define the height function $|\cdot| : X \to \omega$ on the union $X = \bigcup_{n\in\omega} X_n$ by the formula

$$|x| = \min\{n \in \omega : x \in X_n\}.$$

Theorem 1. *For a closed tower $(X_n)_{n\in\omega}$ of uniform spaces the family*

$$\mathcal{B} = \big\{B(x, \textstyle\sum_{i\ge|x|} U_i) : x \in X,\ U_i \in \mathcal{U}_{X_i},\ i \ge |x|\big\}$$

is a base of the topology of the uniform direct limit $\text{u-}\varinjlim X_n$.

This theorem can be used for recognizing the topology of the direct limits in the category of topological groups. Each topological group G will be endowed with the left uniformity generated by the family of entourages $\{(x,y)\in G^2 : x\in yU\}$ where $U=U^{-1}$ runs over symmetric neighborhoods of the neutral element e of G.

Let us say that a triple (G,Γ,H) of topological groups is a *SIN-triple* if for any neighborhoods $V\subset\Gamma$ and $U\subset G$ of the neutral element e of G the product $V\cdot\sqrt[H]{U}$ is a neighborhood of e in G. Here $\sqrt[H]{U}=\{x\in G : x^H\subset U\}$ and $x^H=\{hxh^{-1} : h\in H\}$ is the conjugacy class of a point x.

Theorem 2. *The uniform direct limit* u-$\varinjlim X_n$ *of a closed tower of topological groups* $(X_n)_{n\in\omega}$ *is a topological group provided each triple* (X_{n+1},X_n,X_{n-1}) *is a SIN-triple. In this case* u-$\varinjlim X_n$ *can be identified with the direct limit* g-$\varinjlim X_n$ *of the tower* (X_n) *in the category of topological groups.*

Corollary 1. *The uniform direct limit* u-$\varinjlim X_n$ *of a closed tower of topological SIN-groups* $(X_n)_{n\in\omega}$ *is a topological group, which can be identified with the direct limit* g-$\varinjlim X_n$ *of the tower* (X_n) *in the category of topological groups.*

Corollary 2. *The uniform direct limit* u-$\varinjlim X_n$ *of a closed tower of linear topological spaces* $(X_n)_{n\in\omega}$ *is a linear topological space, which can be identified with the direct limit* l-$\varinjlim X_n$ *of the tower* (X_n) *in the category of linear topological spaces.*

Corollary 3. *The uniform direct limit* u-$\varinjlim X_n$ *of a closed tower of locally convex linear topological spaces* $(X_n)_{n\in\omega}$ *is a locally convex space, which can be identified with the direct limit* lc-$\varinjlim X_n$ *of the tower* (X_n) *in the category of locally convex spaces.*

2. A topological characterization of LF-spaces.

By an LF-space we understand a direct limit lc-$\varinjlim X_n$ of a tower

$$X_0\subset X_1\subset X_2\subset\cdots$$

of Fréchet (=locally convex linear metric) spaces in the category of locally convex spaces. By Corollary 3, the space lc-$\varinjlim X_n$ can be identified with the uniform direct limit u-$\varinjlim X_n$ of the tower (X_n) in the category of uniform spaces.

The simplest LF-space is the direct limit $\mathbb{R}^\infty=$ lc-$\varinjlim\mathbb{R}^n$ of the tower $(\mathbb{R}^n)_{n\in\omega}$ of finite-dimensional Euclidean spaces where each space $\mathbb{R}^n$ is identified with the hyperplace $\mathbb{R}^n\times\{0\}$ in $\mathbb{R}^{n+1}$. The LF-space $\mathbb{R}^\infty$ can be identified with the countable direct sum $\oplus_{n\in\omega}\mathbb{R}$ of real lines in the category of locally convex spaces.

In 1974 P.Mankiewicz obtained a topological clssification of LF-spaces and proved that each LF-space is homeomorphic to one of the following:

- the Hilbert space $l_2(\kappa)$ that has an orthonormal base of cardinality $\kappa\ge 0$;
- the product $l_2(\kappa)\times\mathbb{R}^\infty$ for some cardinal $\kappa\ge\aleph_0$;
- the direct sum $\oplus_{i\in\omega}l_2(\kappa_i)$ for some increasing sequence $(\kappa_i)_{i\in\omega}$ of cardinals.

The topology of the Hilbert spaces $l_2(\kappa)$ has been characterized by H.Toruńczyk in 1981.

Theorem 3 (Toruńczyk)**.** *A topological space* X *is homeomorphic to an infinite-dimensional Hilbert space* $l_2(\kappa)$ *if and only if* X *is a completely-metrizable AR of density* κ *and each map* $f:A\to X$ *from a completely-metrizable space* A *of density* $\mathrm{dens}(A)\le\kappa$ *can be uniformly approximated by closed embeddings.*

This theorem was applied to obtain a topological clasification of Polish ANR-groups.

Corollary 4 (Dobrowolski-Toruńczyk). *A topological group G is (locally) homeomorphic to a separable Hilbert space if and only if G is a Polish [=separable completely-metrizable] absolute (neighborhood) retract.*

The topology of the LF-space $\mathbb{R}^\infty$ has been characterized by K.Sakai in 1984:

Theorem 4 (Sakai). *A topological space X is homeomorphic to the LF-space $\mathbb{R}^\infty$ if and only if*

1. *X is homeomorphic to the topological direct limit $\text{t-}\varinjlim X_n$ of a tower of finite-dimensional metrizable compact spaces;*
2. *each embedding $f : B \to X$ of a closed subspace $B \subset C$ of a finite-dimensional metrizable compactum C extends to an embedding $\bar{f} : C \to X$.*

This theorem has many applications in topological algebra, in particular:

Corollary 5 (Zarichnyi). *The free topological group $F(X)$ of any non-discrete finite-dimensional compact absolute (neighborhood) retract X is (locally) homeomoprhic to $\mathbb{R}^\infty$.*

We are going to use the uniform direct limit for topological characterization of LF-spaces distinct from $l_2(\kappa)$ and $\mathbb{R}^\infty$.

Definition 1. A uniform space X is defined to be *uniformly equiconnected* if there is a continuous map $\lambda : X^2 \times [0,1] \to X$ such that

- $\lambda(x, y, 0) = x$, $\lambda(x, y, 1) = y$ for all $x, y \in X$;
- for every entourage $U \in \mathcal{U}_X$ there is an entourage $V \in \mathcal{U}_X$ such that for each pair $(x, y) \in V$ and any $t, \tau \in [0,1]$ we get $\big(\lambda(x, y, \tau), \lambda(x, y, t)\big) \in U$;

Definition 2. A subset A of a uniform space X is defined to be a *uniform retract* in X if there is a retraction $r : X \to A$ such that for every entourage $U \in \mathcal{U}_X$ there is an entourage $V \in \mathcal{U}_X$ such that $(r(x), x) \in U$ for any point $x \in B(A, V)$.

Definition 3. We say that a subset A of a uniform space X has a *uniform collar in X* if there is a continuous map $e : A \times [0,1] \to X$ such that

- for any entourage $U \in \mathcal{U}_X$ there is a number $\delta > 0$ such that $e(a,t) \in B(a, U)$ for any $(a, t) \in A \times [0, \delta]$;
- for every $\delta \in (0,1]$ there is an entourage $U \in \mathcal{U}_X$ such that $e(a,t) \notin B(A,U)$ for any $(a, t) \in A \times [\delta, 1]$.

Now we are able to formulate our topological characterization of LF-spaces.

Theorem 5. *A topological space X is homeomorphic to a non-metrizable LF-space if and only if X is homeomorphic to the uniform direct limit $\text{u-}\varinjlim X_n$ of a tower $(X_n)_{n\in\omega}$ of metrizable uniform spaces such that each space X_n*

1. *is locally equiconnected;*
2. *is a uniform retract in X_{n+1};*
3. *has a uniform collar in X_{n+1};*
4. *is homeomorphic to a Hilbert space.*

Corollary 6. *The uniform direct limit $\text{u-}\varinjlim X_n$ of a tower $(X_n)_{n\in\omega}$ of Polish AR-groups is homeomorphic to a separable LF-space provided each group X_n is a uniform retract in X_{n+1}.*

Corollary 7. *For each connected non-compact surface M the group $\mathcal{H}_c(M)$ of compactly supported homeomorphisms of M, endowed with the Whitney topology is homeomoprhic to the LF-space $l_2 \times \mathbb{R}^\infty$.*

References

[1] T.Banakh, The topological structure of direct limits in the category of uniform spaces, preprint.
[2] T.Banakh, A topological characterization of LF-spaces, preprint.