CONSTRUCTING SUBSETS OF A GIVEN PACKING INDEX IN ABELIAN GROUPS

N. LYASKOVSKA

Abstract. By definition, the sharp packing index $\mathrm{ind}^\sharp_P(A)$ of a subset A of an abelian group G is the smallest cardinal κ such that for any subset $B\subset G$ of size $|B|\ge\kappa$ the family $\{b+A: b\in B\}$ is not disjoint. We prove that an infinite Abelian group G contains a subset A with given index $\mathrm{ind}^\sharp_P(A)=\kappa$ if and only if one of the following conditions holds: (1) $2\le\kappa\le|G|^+$ and $k\notin\{3,4\}$; (2) $\kappa=3$ and G is not isomorphic to $\oplus_{i\in I}\mathbb{Z}_3$; (3) $\kappa=4$ and G is not isomorphic to $\oplus_{i\in I}\mathbb{Z}_2$ or to $\mathbb{Z}_4\oplus(\oplus_{i\in I}\mathbb{Z}_2)$.

The famous problem of optimal sphere packing traces its history back to B.Pascal and belongs to the most difficult problems of combinatorial geometry [CS]. In this paper we consider an analogous problem in the algebraic setting. Namely, given a subset A of an Abelian group G we study the cardinal number

$$\mathrm{ind}_P(A)=\sup\big\{|B| : B\subset G \text{ and } (B-B)\cap(A-A)=\{0\}\big\}$$

called the *packing index* of A in G. Note that the equality $(B-B)\cap(A-A)=\emptyset$ holds if and only if $(b+A)\cap(b'+A)=\emptyset$ for any distinct points $b,b'\in B$. Therefore, $\mathrm{ind}_P(A)$ can be thought as the maximal number of pairwise disjoint shift copies of A that can be placed in the group G. In this situation it is natural to ask if such a maximal number always exists. In fact, this was a question of D.Dikranjan and I.Protasov who asked in [DP] if for each subset $A\subset\mathbb{Z}$ with $\mathrm{ind}_P(A)\ge\aleph_0$ there exists an infinite family of pairwise disjoint shifts of A. The answer to this problem turned out to be negative, see [BL$_1$], [BL$_2$]. So the supremum in the definition of $\mathrm{ind}_P(A)$ cannot be replaced by the maximum.

To catch the difference between sup and max, let us adjust the definition of the packing index $\mathrm{ind}_P(X)$ and define the cardinal number

$$\mathrm{ind}^\sharp_P(A)=\min\{\kappa : \forall B\subset G\ \ |B|\ge\kappa\Rightarrow(B-B\cap A-A\ne\{0\})\}$$

called the *sharp packing index* of A in G. In terms of the sharp packing index the question of D. Dikranjan and I. Protasov can be reformulated as finding a subset $A\subset\mathbb{Z}$ with $\mathrm{ind}^\sharp_P(A)=\aleph_0$. According to [BL$_2$] (and [BL$_1$]) such a set A can be found in each infinite (abelian) group G. Having in mind this result, I.Protasov asked in a private conversation if for any non-zero cardinal $\kappa\le|G|$ there is a set $A\subset G$ with $\mathrm{ind}_P(A)=\kappa$. In this paper we answer this question affirmatively (with three exceptions). Firstly, we treat a similar question for the sharp packing index because its value completely determines the value of $\mathrm{ind}_P(A)$:

$$\mathrm{ind}_P(A)=\sup\{\kappa:\kappa<\mathrm{ind}^\sharp_P(A)\}.$$

Our principal result is

Main Theorem. *An infinite Abelian group G contains a subset $A\subset G$ with sharp packing index $\mathrm{ind}^\sharp_P(A)=\kappa$ if and only if one of the following conditions holds:*

1991 *Mathematics Subject Classification.* 20K99, 05D05.

Key words and phrases. packing ingex, group.

1) $2 \le \kappa \le |G|^+$ *and* $\kappa \notin \{3, 4\}$.
2) $\kappa = 3$ *and* G *is not isomorphic to* $\oplus_{i \in I} \mathbb{Z}_3$.
3) $\kappa = 4$ *and* G *is not isomorphic to* $\oplus_{i \in I} \mathbb{Z}_2$ *or to* $\mathbb{Z}_4 \oplus (\oplus_{i \in I} \mathbb{Z}_2)$.

Using the relation between the packing and sharp packing indices, we can derive from the above theorem an analogous characterization of possible values of the packing index.

Corollary. *An infinite Abelian group* G *contains a subset* $A \subset G$ *with packing index* $ind_P(A) = \kappa$ *if and only if one of the following conditions holds:*

1) $1 \le \kappa \le |G|$ *and* $\kappa \notin \{2, 3\}$.
2) $\kappa = 2$ *and* G *is not isomorphic to* $\oplus_{i \in I} \mathbb{Z}_3$.
3) $\kappa = 3$ *and* G *is not isomorphic to* $\oplus_{i \in I} \mathbb{Z}_2$ *or to* $\mathbb{Z}_4 \oplus (\oplus_{i \in I} \mathbb{Z}_2)$.

1. Preliminaries

In the proof of Main Theorem we shall exploit a combinatorial lemma proved in this section. For a set A by $[A]^2 = \{B \subset A : |B| = 2\}$ we denote the family of all two-element subsets of A.

We shall say that a map $f : [A]^2 \mapsto [B]^2$

- is *separately injective* if for any $a \in A$ the map $f_a : x \mapsto f(\{x, a\})$ is injective;
- *preserves intersections* if for any $a_0, a_1, a_2 \in A$ the intersection $f(\{a_0, a_1\}) \cap f(\{a_0, a_2\})$ is not empty.

Lemma 1. *If* $|A| \ge 5$ *and a map* $f : [A]^2 \mapsto [B]^2$ *is separately injective and preserves intersections, then* $|A| \le |B|$.

Proof. Fix any point $a_0 \in A$ and consider the family $\{f(\{a, a_0\}) : a \in A \backslash \{a_0\}\}$. Since f preserves intersections we have that $f(\{a, a_0\}) \cap f(\{a', a_0\}) \ne \emptyset$ for any distinct $a, a' \in A$. Using the separately injective of f and the inequality $|A| \ge 5$ we can prove that the intersection $\bigcap_{a \in A \backslash \{a_0\}} f(\{a, a_0\})$ is not empty and hence contains some element b_0. Thus we obtain that $f : \{a, a_0\} \mapsto \{b, b_0\}$. And since f is separately injective we obtain an injective map from $A \backslash \{a_0\}$ into $B \backslash \{b_0\}$ implying the desired inequality $|A| \le |B|$. □

We shall also need one structure property of Abelian groups. By $\mathbb{Z}$ we denote the additive group of integer numbers and by

$$\mathbb{Z}(p^\infty) = \{z \in \mathbb{C} : \exists n \in \mathbb{N} \text{ with } z^{p^k} = 1\}$$

the quasicyclic p-group for a prime number p.

Proposition 1. *Each infinite Abelian group* G *contains an infinite subgroup isomorphic to* $\mathbb{Z}$, $\mathbb{Z}(p^\infty)$ *or the direct sum of finite cyclic groups.*

Proof. If G contains an element g of infinite order, then it generates a cyclic subgroup isomorphic to $\mathbb{Z}$. Otherwise, H is a torsion group and by Theorem 8.4 [Fu] decomposes into the direct sum $G = \oplus_p A_p$ of p-groups A_p. If each group A_p is finite, then G contains an infinite direct product of finite cyclic group. If for some prime number p the p-group A_p is infinite, then there are two cases. Either A_p contains a copy of the quasicyclic p-group $\mathbb{Z}(p^\infty)$ or else each element of A_p has finite height. In the latter case, take any infinite countable subgroup $H \subset A_p$ and apply Theorem 17.3 of [Fu] to conclude that H is the direct sum of finite cyclic groups. □

2. The proof of the "only if" part of Main Theorem

The proof of the "only if" part of Main Theorem is divided into two lemmas.

Lemma 2. *If a group G contains a subset $A \subset G$ with $ind_p^\sharp(A) = 3$ (which is equivalent to* $\mathrm{ind}_P(A) = 2$*), then G is not isomorphic to the direct sum $\oplus_{i\in I}\mathbb{Z}_3$.*

Proof. On the contrary suppose that G is isomorphic to the direct sum $\oplus_{i\in I}\mathbb{Z}_3$ and take a subset A of G with $\mathrm{ind}_P^\sharp(A) = 3$. The latter is equivalent to $\mathrm{ind}_P(A) = 2$ which means that there is a subset $B_2 \subset G$ of size 2 such that the family $\{b + A : b \in B_2\}$ is disjoint. Note that for every $b' \in G$ the family $\{b + A : b \in b' + B_2\}$ is disjoint too. So without loss of generality we can assume that $B_2 = \{0, b_1\}$. The family $\{b + A : b \in B_2\}$ is disjoint and hence

$$A \cap (b_1 + A) = \emptyset.$$

Adding to both sides b_1 and $2b_1$ we get

$$(b_1 + A) \cap (2b_1 + A) = \emptyset;$$

$$(2b_1 + A) \cap (3b_1 + A) = \emptyset.$$

Since G is isomorphic to the direct sum $\oplus_{i\in I}\mathbb{Z}_3$ we get $3b_1 = 0$. Thus we conclude that $\big\{b + A : b \in \{0, b_1, 2b_1\}\big\}$ is disjoint and so $\mathrm{ind}_P(a) > 2$ and $ind_p^\sharp(A) > 3$, which contradicts our assumption. □

Lemma 3. *If a group G contains a subset $A \subset G$ with $ind_p^\sharp(A) = 4$ (which is equivalent to* $\mathrm{ind}_P(A) = 3$*), then G can not be isomorphic neither to the direct sum $\oplus_{i\in I}\mathbb{Z}_2$ nor to $\mathbb{Z}_4 \oplus (\oplus_{i\in I}\mathbb{Z}_2)$.*

Proof. Conversely suppose that G is isomorphic to $\oplus_{i\in I}\mathbb{Z}_2$ or to $\mathbb{Z}_4 \oplus (\oplus_{i\in I}\mathbb{Z}_2)$ and there exists a subset A of G with $\mathrm{ind}_P^\sharp(A) = 4$. This is equivalent to $\mathrm{ind}_P(A) = 3$ and from the definition we get that there is a three-element subset $B_3 \subset G$ such that the family $\{b + A : b \in B_3\}$ is disjoint. Note that for any $b' \in G$ the family $\{b + A : b \in b' + B_3\}$ is disjoint too. So, without loss of generality we can assume that $B_3 = \{0, b_1, b_2\}$. Since the family $\{b + A : b \in B_3\}$ is disjoint we conclude that

$$A \cap (b_1 + A) = \emptyset; \tag{1}$$
$$A \cap (b_2 + A) = \emptyset; \tag{2}$$
$$(b_1 + A) \cap (b_2 + A) = \emptyset. \tag{3}$$

We consider three cases.

Case 1. Suppose one of the elements b_1, b_2 is of order 2. Let it be b_1. Then $2b_1 = 0$ and

$$(2) + b_1: \quad (b_1 + A) \cap (b_1 + b_2 + A) = \emptyset;$$
$$(3) + b_1: \quad A \cap (b_1 + b_2 + A) = \emptyset.$$
$$(1) + b_2: \quad (b_2 + A) \cap (b_2 + b_1 + A) = \emptyset.$$

Thus we get that the family $\big\{b + A : b \in \{0, b_1, b_2, b_1 + b_2\}\big\}$ is disjoint and hence $\mathrm{ind}_P(A) > 3$ and $\mathrm{ind}_p^\sharp(A) > 4$, which contradicts our assumption. Thus we complete the proof of the Case 1.

Next we consider two cases where both b_1 and b_2 are of order 4. In this case the group G is isomorphic to $\mathbb{Z}_4 \oplus (\oplus_{i\in I}\mathbb{Z}_2)$. Therefore there are two possibilities: $b_1 = (g, x), b_2 = (g, y)$ or $b_1 = (g, x), b_2 = (-g, y)$ where $x, y \in \oplus_{i\in I}\mathbb{Z}_2$ and $g \in \mathbb{Z}_4$ is of order 4.

Case 2. Suppose $b_1 = (g, x), b_2 = (g, y)$ where $x, y \in \oplus_{i\in I}\mathbb{Z}_2$ and $g \in \mathbb{Z}_4$ is of order 4.

Recall that $B_3=\{(0,0),(g,x),(g,y)\}$ and consider the set $B_4=\{(0,0),(g,x),(g,y),(0,x+y)\}$. We claim that the family $\{b+A:b\in B_4\}$ is disjoint. Indeed, since $\{b+A:b\in B_3\}$ is disjoint we have:

$$\begin{array}{ll}(1) & A\cap((g,x)+A)=\emptyset;\\ (2) & A\cap((g,y)+A)=\emptyset;\\ (3) & ((g,x)+A)\cap((g,y)+A)=\emptyset.\end{array}$$

Then

$$\begin{array}{lll}(3)+(3g,y) & : & ((0,x+y)+A)\cap A=\emptyset;\\ (2)+(0,x+y) & : & ((0,x+y)+A)\cap((g,x)+A)=\emptyset;\\ (1)+(0,x+y) & : & ((0,x+y)+A)\cap((g,y)+A)=\emptyset.\end{array}$$

Hence, the family $\{b+A:b\in B_4\}$ is disjoint which implies $\mathrm{ind}_P(A)\geq 3$ and $\mathrm{ind}_P^\sharp(A)\geq 4$, a contradiction with the assumption.

Case 3. Suppose $b_1=(g,x),b_2=(-g,y)$ where $x,y\in\oplus_{i\in I}\mathbb{Z}_2$ and $g\in\mathbb{Z}_4$ is of order 4.

In this case $B_3=\{(0,0),(g,x),(-g,y)\}$. Put $B_4=\{(0,0),(g,x),(-g,y),(2g,x+y)\}$. We claim that the family $\{b+A:b\in B_4\}$ is disjoint. Indeed, since $\{b+A:b\in B_3\}$ is disjoint we have:

$$\begin{array}{ll}(1) & A\cap((g,x)+A)=\emptyset;\\ (2) & A\cap((-g,y)+A)=\emptyset;\\ (3) & ((g,x)+A)\cap((-g,y)+A)=\emptyset.\end{array}$$

Then

$$\begin{array}{lll}(3)+(g,y) & : & ((2g,x+y)+A)\cap A=\emptyset;\\ (2)+(2g,x+y) & : & ((2g,x+y)+A)\cap((g,x)+A)=\emptyset;\\ (1)+(2g,x+y) & : & ((2g,x+y)+A)\cap((-g,y)+A)=\emptyset.\end{array}$$

Hence the family $\{b+A:b\in B_4\}$ is disjoint and thus $\mathrm{ind}_P(A)>3$ and $\mathrm{ind}_P^\sharp(A)>4$, which contradicts our assumption. ☐

Thus if G contains a subset $A\subset G$ with $\mathrm{ind}_P^+(A)=\kappa$ then one of the condition 1)-3) holds.

3. The proof of the "if" part of Main Theorem

To prove the "if" part of the Main Theorem, given a cardinal κ satisfying one of the conditions 1)–3) we shall construct a subset A with $\mathrm{ind}_P^\sharp(A)=\kappa$. First we shall construct a subset A_κ assuming that we have in disposal an auxiliary subset $\mathbb{B}_\kappa$ with some properties. Next, a subset $\mathbb{B}_\kappa$ wil the desired properties will be constructed in each group.

Proposition 2. *An infinite Abelian group G contains a subset A_κ with $\mathrm{ind}_P^\sharp(A_\kappa)=\kappa$ if there exists a subset $\mathbb{B}_\kappa=-\mathbb{B}_\kappa$ of G with the following properties:*

(1_κ) *for every cardinal $\alpha<\kappa$ there is a subset B_α of size $|B_\alpha|=\alpha$ such that $B_\alpha-B_\alpha\subset\mathbb{B}_\kappa$;*
(2_κ) *$B_\kappa-B_\kappa\not\subset\mathbb{B}_\kappa$ for any subset $B_\kappa\subset G$ of size κ;*
(3_κ) *$F+\mathbb{B}_\kappa\neq G$ for any subset $F\subset G$ of size $|F|<|G|$.*

By $|A|$ we denote the cardinality of a set A.

Proof. Let $\mathbb{B}_\kappa^\circ=\mathbb{B}_\kappa\setminus\{0\}$. We shall construct a subset $A_\kappa\subset G$ such that $(\mathbb{B}_\kappa^\circ+A_\kappa)\cap A_\kappa=\emptyset$. Moreover, the subset A_κ will be constructed so that $G\backslash\mathbb{B}_\kappa^\circ\subset A_\kappa-A_\kappa$.

Let $\lambda=|G\backslash\mathbb{B}_\kappa^\circ|$ and $G\backslash\mathbb{B}_\kappa^\circ=\{g_\alpha:\alpha<\lambda\}$ be an enumeration of $G\backslash\mathbb{B}_\kappa^\circ$ by ordinals $\alpha<\lambda$.

We put $A_\kappa=\bigcup_{\alpha<\lambda}\{a_\alpha,g_\alpha+a_\alpha\}$, where a sequence $(a_\alpha)_{\alpha<\lambda}$ is to be defined later. This clearly forces that $G\backslash\mathbb{B}_\kappa^\circ\subset A_\kappa-A_\kappa$.

The task is now to find a sequence $(a_\alpha)_{\alpha<\lambda}$ such that $(\mathbb{B}_\kappa^\circ + A_\kappa) \cap A_\kappa = \emptyset$. We define this sequence by induction.

We start with $a_0 = 0$. Assuming that for some α the points $a_\beta, \beta < \alpha$, have been constructed, put $F_\alpha = \{a_\beta, g_\beta + a_\beta : \beta < \alpha\}$.

According to the property (3_κ) of the set $\mathbb{B}_\kappa$ we can pick a point $a_\alpha \in G$ so that

$$a_\alpha \notin F_\alpha + \mathbb{B}_\kappa \cup F_\alpha - g_\alpha + \mathbb{B}_\kappa.$$

This gives $(\mathbb{B}_\kappa^\circ + A_\kappa) \cap A_\kappa = \emptyset$.

It remains to show that A_κ satisfies the conclusion of the theorem.

According to the property (1_κ) of the set $\mathbb{B}_\kappa$ for any cardinal $\alpha < \kappa$ there is B_α such that $B_\alpha - B_\alpha \subset \mathbb{B}_\kappa$. From the fact that $\mathbb{B}_\kappa^\circ + A_\kappa \cap A_\kappa = \emptyset$ we conclude that $b - b' + A_\kappa \bigcap A_\kappa = \emptyset$ for all distinct $b, b' \in B_\alpha$. Thus for any cardinal $\alpha < \kappa$ there is B_α such that the family $\{b + A_\kappa : b \in B_\alpha\}$ is disjoint and so $\mathrm{ind}_P^\sharp(A_\kappa) \geq \kappa$.

Let us show that $\mathrm{ind}_P^\sharp(A_\kappa) = \kappa$. According to the property (2_κ), for any subset $B_\kappa \subset G$ of size κ there are $b, b' \in B_\kappa$ such that $b - b' \notin \mathbb{B}_\kappa$. Therefore $b - b' \in G \backslash \mathbb{B}_\kappa^\circ \subset A_\kappa - A_\kappa$. Hence $b + A_\kappa \bigcap b' + A_\kappa \neq \emptyset$, which yields $\mathrm{ind}_P^\sharp(A_\kappa) \leq \kappa$. Combining the two inequalities, we get $\mathrm{ind}_P^\sharp(A_\kappa) = \kappa$. □

The proof of the Main Theorem will be completed as soon as we construct a subset $\mathbb{B}_\kappa$ with properties $(1_\kappa) - (3_\kappa)$. This will be done in the following five lemmas.

Lemma 4. *Let $\kappa = 3$ and G be an infinite Abelian group which is not isomorphic to the direct sum $\oplus_{i\in I}\mathbb{Z}_3$. Then G contains a subset $\mathbb{B}_3$ with the properties* $(1_3) - (3_3)$.

Proof. Pick any nonzero point $g \in G$ whose order is not equal to 3 and consider the set $\mathbb{B}_3 = B_2 - B_2 = \{0, \pm g\}$ where $B_2 = \{0, g\}$. It is clear that $\mathbb{B}_3$ has the properties (1_3), (3_3). So it is enough to show that $\mathbb{B}_3$ satisfies the property (2_3). Note that if $2g = 0$ then $\mathbb{B}_3 = \{0, g\}$ is a subgroup of G and hence has the property (2_3).

So we assume that $2g \neq 0$ which yields that $\mathbb{B}_3 = \{0, g, -g\}$ contains three elements. To prove that $\mathbb{B}_3$ has property (2_3) fix some subset $B_3 \subset G$ of size 3 and pick any point $b_0 \in B_3$. If there is $b \in B_3$ with

$$b - b_0 \notin \mathbb{B}_3 = \{0, g, -g\}$$

then there is nothing to prove. Otherwise we have that

$$B_3 - b_0 \subset \mathbb{B}_3.$$

Since $|B_3| = 3$ there are $b, b' \in B_3$ such that $b - b_0 = g$; $b' - b_0 = -g$. Hence we get $b - b' = 2g$. From the choice of element g we get that $2g \notin \mathbb{B}_3$. Hence $b_2 - b_3 \notin \mathbb{B}_3$ and $\mathbb{B}$ has the property (2_3) which completes the proof of the lemma. □

Lemma 5. *Let $\kappa = 4$ and G be an infinite Abelian which is not isomorphic to $\oplus_{i\in I}\mathbb{Z}_2$ or to $\mathbb{Z}_4 \oplus (\oplus_{i\in I}\mathbb{Z}_2)$. Then G contains a subset $\mathbb{B}_4$ with properties* $(1_4) - (3_4)$.

Proof. We consider three cases.

Case 1. Suppose a group G contains an element g with order > 5.

Put $\mathbb{B}_4 = B_3 - B_3 = \{0, \pm g, \pm 2g\}$ where $B_3 = \{0, g, -g\}$. It is easily to check that $\mathbb{B}_4$ has the properties $(1_4), (3_4)$. We claim that $\mathbb{B}_4$ satisfies the property (2_4).

To derive a contradiction, suppose that there is a subset $B_4 \subset G$ of size $|B_4| = 4$ such that $B_4 - B_4 \subset \mathbb{B}_4 = \{0, g, -g, 2g, -2g\}$.

Fix some element $b_0 \in B_4$. Since $B_4 - b_0 \subset \mathbb{B}_4$ there are $b, b' \in B_4$ such that
$b - b_0 = -g; b' - b_0 = 2g$ or $b - b_0 = g; b' - b_0 = -2g$.
Then $b' - b = 3g$ or $b' - b = -3g$.

Note that since the order of g is greater than 5, neither $3g \in \mathbb{B}_4$ no $-3g \in \mathbb{B}_4$. Thus we get $b' - b \notin \mathbb{B}_4$, a contradiction with the assumption. Hence $\mathbb{B}_4$ satisfies the property (2_4) and we complete the proof of Case 1.

Case 2. Assume that G contains no element of order greater than 5. Then G is the direct sum of cyclic groups according to Theorem 17.2 of [Fu]. More precisely, G is isomorphic either to $(\oplus_{i\in I}\mathbb{Z}_2) \oplus (\oplus_{j\in J}\mathbb{Z}_4)$ or to $\oplus_{i\in I}\mathbb{Z}_3$ or to $\oplus_{i\in I}\mathbb{Z}_5$. Since G is not isomorphic to $\oplus_{i\in I}\mathbb{Z}_2$ or $\mathbb{Z}_4 \oplus \oplus_{i\in I}\mathbb{Z}_2$, we have to consider the following two cases: G contains a subgroup isomorphic to $\mathbb{Z}_3$ and G is contains a subgroup isomorphic to $\mathbb{Z}_i \oplus \mathbb{Z}_j \oplus H$ for some $4 \le i, j \le 5$.

Case 2a. Suppose that G contains a subgroup H isomorphic to $\mathbb{Z}_3$.

In this sace we put $\mathbb{B}_4 = H$ and see that $\mathbb{B}_4$ has the properties $(1_4) - (3_4)$.

Case 2b. Suppose G contains a subgroup isomorphic to the direct sum of $\mathbb{Z}_i \oplus \mathbb{Z}_j \oplus H$ for some $4 \le i, j \le 5$.

We shall identify $\mathbb{Z}_i \oplus \mathbb{Z}_j$ with a subgroup of G and shall find a subset $\mathbb{B}_4 \subset \mathbb{Z}_i \oplus \mathbb{Z}_j$ with the properties $(1_4) - (3_4)$. Obviously $\mathbb{B}_4$ has the same properties in the whole group G.

Put $\mathbb{B}_4 = B_3 - B_3$ where $B_3 = \{(0,0), (g_1, 0), (0, g_2)\}$. It is clear that $\mathbb{B}_4$ has the properties $(1_4), (3_4)$. We claim that $\mathbb{B}_4$ has property (2_4). Indeed, assuming the converse, we would find a subset $B_4 \subset G$ of size $|B_4| = 4$ with $B_4 - B_4 \subset B_3 - B_3$.

Fix any point $b_0 \in B_4$. Then

$$B_4 - b_0 \subset \mathbb{B}_4 = \{(0,0), (g_1,0), (0,g_2), (-g_1,0), (0,-g_2), (g_1,-g_2), (-g_1,g_2)\}.$$

Let us show that $(g_1, 0) \notin B_4 - b_0$. Since the elements g_1 and g_2 have order ≥ 4,

$$(g_1,0) - (-g_1,0) \notin \mathbb{B}_4;$$
$$(g_1,0) - (-g_1,g_2) \notin \mathbb{B}_4;$$
$$(g_1,0) - (0,-g_2) \notin \mathbb{B}_4.$$

Thus if there is $b \in B_4$ with $b - b_0 = (g_1, 0)$ then

$$B_4 - b_0 \subset \mathbb{B}_4 = \{(0,0), (g_1,0), (0,g_2), (g_1,-g_2)\}.$$

From the above and the fact that $|B_4| = 4$ we get that there are $b_1, b_2 \in B_4$ such that $b_1 - b_0 = (0, g_2)$ and $b_2 - b_0 = (g_1, -g_2)$. Hence $b_2 - b_1 = (g_1, -2g_2) \notin \mathbb{B}_4$, a contradiction with the assumption that $B_4 - B_4 \subset \mathbb{B}_4$. So, we conclude that $(g_1, 0) \notin B_4 - b_0$.

In the same manner we can show that none of the elements $(0, g_2), (-g_1, 0), (0, -g_2)$ belong to $B_4 - b_0$, which contradicts the fact that $B_4 - B_4 \subset \mathbb{B}_4$. This completes the proof of Lemma. □

Lemma 6. *If $\kappa > 4$ is a finite cardinal, then each infinite Abelian group Gcontains a subset $\mathbb{B}_\kappa$ with the properties $(1_\kappa) - (3_\kappa)$.*

Proof. It is easy to check that each subset $\mathbb{B}_\kappa$ with the properties (1_κ)–(3_κ) in a subgroup $H \subset G$ has these properties in the whole group G. This observation combined with Proposition 1 reduces the problem to constructing a set $\mathbb{B}_\kappa$ in the groups $\mathbb{Z}$, $\mathbb{Z}(p^\infty)$ or the direct sum of finite cyclic groups. This will be done separately in the following three cases.

Case 1. We construct a subset $\mathbb{B}_k$ in the group $\mathbb{Z}$.

In this case put $\mathbb{B}_\kappa = B_{\kappa-1} - B_{\kappa-1}$ where $B_{\kappa-1} = \{i : 1 \le i \le \kappa - 1\}$. It is easy to check that $\mathbb{B}_\kappa$ has property $(1_\kappa) - (3_\kappa)$ in $\mathbb{Z}$.

Case 2. We construct a subset $\mathbb{B}_\kappa$ in the quasicyclic p-group $\mathbb{Z}(p^\infty)$.

Choose n such that $z^{p^n} \in \{e^{i\phi} : \frac{2\pi}{\kappa} < \phi < \frac{2\pi}{\kappa-1}\}$. Then put $\mathbb{B}_\kappa = B_{\kappa-1} - B_{\kappa-1}$ where

$$B_{\kappa-1} = \{e^{i\phi} : \phi = \frac{2\pi l}{p^n},\ 1 \le l \le \kappa - 1\}.$$

It is easy to check that $\mathbb{B}_\kappa$ has the properties $(1_\kappa) - (3_\kappa)$ in $\mathbb{Z}(p^\infty)$.

Case 3. We construct a subset $\mathbb{B}_\kappa$ in the direct sum of cyclic groups $\oplus_{i\in\omega}\langle g_i\rangle$.

Put $\mathbb{B}_\kappa = B_{\kappa-1} - B_{\kappa-1}$ where $B_{\kappa-1} = \{g_i : 1 \le i \le \kappa - 1\}$. Obviously $\mathbb{B}_\kappa$ has properties $(1_\kappa), (3_\kappa)$. We claim that $\mathbb{B}_\kappa$ has property (2_κ). To obtain a contradiction assume that there exists a subset $B_\kappa \subset G$ with size $|B_\kappa| = \kappa$ such that

$$B_\kappa - B_\kappa \subset B_{\kappa-1} - B_{\kappa-1}.$$

Consider the sets $S = \{i : 1 \le i \le \kappa - 1\}$ and $F = \{i : 1 \le i \le \kappa\}$. We can enumerate the sets $B_{\kappa-1}$ and B_κ as $B_{\kappa-1} = \{g_i : i \in S\}$ and $B_\kappa = \{b_i : i \in F\}$.

Since $B_\kappa - B_\kappa \subset B_{\kappa-1} - B_{\kappa-1}$ we can define a map $f : [F]^2 \mapsto [S]^2$ assigning to each pair $\{i, j\} \in [F]^2$ a unique pair $\{k, l\} \in [S]^2$ such that $b_i - b_j = \pm(g_k - g_l)$. A desired contradiction will follow from Lemma 1 as soon as we check that f is separately injective and preserves intersections.

Claim 1. The map f preserves intersections.

To derive a contradiction, suppose that there are distinct $i, i^{'} \in F$ and $j \in F$ such that

$$f(\{i, j\}) \cap f(\{i^{'}, j\}) = \emptyset.$$

Then

$$b_i - b_j = g_k - g_l \text{ and } b_{i'} - b_j = g_n - g_m$$

where k, l, n, m are pairwise distinct.

Hence $b_{i'} - b_i = g_n - g_m - g_k + g_l \notin B_{\kappa-1} - B_{\kappa-1}$, which contradicts the assumption that $B_\kappa - B_\kappa \subset B_{\kappa-1} - B_{\kappa-1}$.

Claim 2. The map f is separately injective.

To derive a contradiction, suppose that there are distinct $i, i^{'} \in F$ and $j \in F$ such that

$$f(\{i, j\}) = f(\{i^{'}, j\}) = \{k, l\}.$$

Since $b_i, b_{i'}$ are distinct we get

$$b_i - b_j = g_k - g_l \text{ and } b_{i'} - b_j = g_l - g_k$$

and thus $b_{i'} - b_i = 2(g_l - g_k) \ne 0$. Note that $2(g_l - g_k) \in B_{\kappa-1} - B_{\kappa-1}$ iff $2(g_l - g_k) = g_k - g_l$. Thus we get that $3g_k = 0$ and $3g_l = 0$.

Since $|F| = \kappa > 4$ we can chose $r \in F\backslash\{i, i^{'}, j\}$. The map f preserves intersections so $f(\{r, j\}) \cap \{k, l\} \ne \emptyset$. Also note that $f(\{r, j\}) \cap \{k, l\} \ne \{k, l\}$ otherwise $b_r = b_i$ or $b_r = b_{i'}$. So without loss of generality we can assume that $f(\{r, j\}) \cap \{k, l\} = \{k\}$.

Hence $b_r - b_j = g_s - g_k$ or $b_r - b_j = g_k - g_s$ for some s.

Consequently, $b_r - b_i = g_s - 2g_k + g_l$ or $b_r - b_{i'} = 2g_k - g_s - g_l$.

Note that $2g_k \ne 0$ since $3g_k = 0$.

So we get $b_r - b_i = g_s - 2g_k + g_l \notin B_{\kappa-1} - B_{\kappa-1}$ or $b_r - b_{i'} = 2g_k - g_s - g_l \notin B_{\kappa-1} - B_{\kappa-1}$. This contradicts the assumption that $B_\kappa - B_\kappa \subset B_{\kappa-1} - B_{\kappa-1}$. □

Lemma 7. *Let κ be an infinite not limit cardinal with $\kappa \le |G|$ where G is an infinite Abelian group. Then there exists a subset $\mathbb{B}_\kappa$ with the properties $(1_\kappa) - (3_\kappa)$.*

Proof. Since κ is infinite not limit cardinal there exists cardinal α such that $\kappa = \alpha^+$. Put $\mathbb{B}_\kappa = B_\alpha - B_\alpha$ where B_α is any subset of G with size $|B_\alpha| = \alpha$. Obviously $\mathbb{B}_\kappa$ satisfies property (1_κ).

Since $|\mathbb{B}_\kappa| = \alpha$ and $|B_\kappa - B_\kappa| = \kappa = \alpha^+$ for any subset $B_\kappa \subset G$ of size κ we get $B_\kappa - B_\kappa \not\subset \mathbb{B}_\kappa$. Therefore $\mathbb{B}_\kappa$ has property (2_κ).

The last property (3_κ) follows from the fact $|F| + |\mathbb{B}_\kappa| \le |F| \cdot |\mathbb{B}_\kappa| < |G|$. □

Lemma 8. *Let κ be a limit cardinal and G be an infinite Abelian group with $\kappa \le |G|$. Then there exists a subset $\mathbb{B}_\kappa \subset G$ with the properties $(1_\kappa) - (3_\kappa)$.*

Proof. Note that it is enough to show that each group G of size κ contains a subset $\mathbb{B}_\kappa$ with properties $(1_\kappa) - (3_\kappa)$. When $|G| > \kappa$ then we can take any subgroup $H \subset G$ of size $|H| = \kappa$ and find a subset $\mathbb{B}_\kappa$ of H with properties $(1_\kappa) - (3_\kappa)$ in H. Then the subset $\mathbb{B}_\kappa$ will have the properties $(1_\kappa) - (3_\kappa)$ in the whole group.

So it remains to prove that such a set $\mathbb{B}_\kappa$ exists in each group G of size κ.

First we describe a sequence of symmetric subsets $F_\alpha \subset G$ of size α such that $G = \bigcup_{\alpha<\kappa} F_\alpha$ and $F_\alpha \supset \bigcup_{\beta<\alpha} F_\beta$. Enumerate the group G so that $G = \{g_\alpha : \alpha < \kappa\}$ and $g_0 = e$. Then put $F_\alpha = \{g_\beta, -g_\beta : \beta < \alpha\}$ for all $\alpha < \kappa$.

We put

$$\mathbb{B}_\kappa = \bigcup_{\alpha<\kappa} B_\alpha - B_\alpha$$

where a set $B_\alpha = \{b_\alpha^\beta : \beta < \alpha\} \subset G$ of size α will be chosen later.

To simplify notation we write $\mathbb{B}_{<\alpha}$ instead of $\bigcup_{\beta<\alpha}(B_\beta - B_\beta)$ and $\mathbb{B}_{>\alpha}$ instead of $\bigcup_{\alpha\le\beta<\kappa}(B_\beta - B_\beta)$. By $B_\alpha^{<\beta}$ we shall denote the initial interval $\{b_\alpha^\gamma : \gamma < \beta\}$ of B_α.

Now we are in a position to define a sequence of sets B_α forcing the set $\mathbb{B}_\kappa$ to satisfy the properties (2_κ) and (3_κ). To ensure property (3_κ) we will also construct a transfinite sequence of points $(h_\alpha)_{\alpha<\kappa}$ of G such that $h_\alpha \notin F_\alpha + \mathbb{B}_\kappa$.

We start putting $B_0 = \{e\}$ and taking any non-zero point $h_0 \in G$. Assume that for some ordinal $\alpha < \kappa$ the sets B_β and the points h_β, $\beta < \alpha$, have been constructed. Then pick any point $h_\alpha \in G$ with

$$h_\alpha \notin F_\alpha + \mathbb{B}_{<\alpha}.$$

Such a point exists because the size of the set $F_\alpha + \mathbb{B}_{<\alpha}$ is equals $\alpha < \kappa = |G|$. Let

$$H_\alpha = \{h_\beta, -h_\beta : \beta \le \alpha\}.$$

Next we define inductively elements of $B_\alpha = \{b_\alpha^\beta : \beta < \alpha\}$.

We pick any b_α^0 with $b_\alpha^0 \in G\backslash\mathbb{B}_{<\alpha}$. Next, we chose b_α^β with

(a) $b_\alpha^\beta \notin B_\alpha^{<\beta} + F_\alpha + \mathbb{B}_{<\alpha}$;
(b) $b_\alpha^\beta \notin B_\alpha^{<\beta} - B_\alpha^{<\beta} + B_\alpha^{<\beta} + F_\alpha$;
(c) $b_\alpha^\beta \notin B_\alpha^{<\beta} + F_\alpha + H_\alpha$.

To ensure properties (a),(b),(c) we have to avoid the sets of size α, which is possible because $|G| = \kappa$.

Now let us prove that the constructed set $\mathbb{B}_\kappa$ satisfies the properties $(1_\kappa)-(3_\kappa)$. In fact, the property (1_κ) is evident while (3_κ) follows immediately from (c). It remains to prove

Claim. *The set $\mathbb{B}_\kappa$ has property* (2κ).

Let B_κ be a subset of G of size $|B_\kappa|=\kappa$. Fix any pairwise distinct points $c_1,c_2,c_3\in B_\kappa$.

If $B_\kappa-B_\kappa\subset\mathbb{B}_\kappa$ then $B_\kappa\subset\bigcap_{i=1}^3 c_i+\mathbb{B}_\kappa$ and $\kappa=|B_\kappa|\le|\bigcap_{i=1}^3 c_i+\mathbb{B}_\kappa|$.

So to prove our claim it is enough to show that $|\bigcap_{i=1}^3 c_i+\mathbb{B}_\kappa|<\kappa$. Find an ordinal $\alpha<\kappa$ such that $c_p-c_q\in F_\alpha$ for any $1\le p,q\le 3$. Assuming that $|\bigcap_{i=1}^3 c_i+\mathbb{B}_\kappa|=\kappa$ we may find a point $b\in\bigcap_{i=1}^3(c_i+\mathbb{B}_{>\alpha})\backslash\{c_i\}$. A contradiction will be reached in three steps.

Step 1. *First show that there is $\beta>\alpha$ with $b\in\bigcap_{i=1}^3(c_i+B_\beta)$.*

Otherwise, $b-c_p\in B_\gamma-B_\gamma$ and $b-c_q\in B_\beta-B_\beta$ for some $\gamma>\beta>\alpha$ and some $p\ne q$. Find $i,j<\gamma$ with $b-c_p=b^i_\gamma-b^j_\gamma$. The inequality $b\ne c_p$ implies $i\ne j$.

If $i<j$ then $b^j_\gamma=b^i_\gamma-b+c_p=b^i_\gamma-b+c_q-c_q+c_p\subset b^i_\gamma-B_\beta+B_\beta+F_\gamma\subset B^{<j}_\gamma+\mathbb{B}_{<\gamma}+F_\gamma$, which contradicts (a).

If $i>j$ then $b^i_\gamma=b^j_\gamma+b-c_p=b^i_\gamma+B_\beta-B_\beta+c_q-c_p\subset B^{<j}_\gamma+\mathbb{B}_{<\gamma}+F_\gamma$, which again contradicts (a).

Step 2. *We claim that if $b-c_p=b^i_\beta-b^j_\beta$ and $b-c_q=b^s_\beta-b^t_\beta$ then* $\max\{i,j\}=\max\{s,t\}$.

It follows from the hypothesis that $c_q-c_p=b^i_\beta-b^j_\beta+b^t_\beta-b^s_\beta$. To obtain a contradiction assume that $\max\{i,j\}>\max\{s,t\}$.

If $j<i$ then $b^i_\beta=c_q-c_p+b^j_\beta-b^t_\beta+b^s_\beta\in F_\beta+B^{<i}_\beta-B^{<i}_\beta+B^{<i}_\beta$, which contradicts (b).

If $i<j$ then $b^j_\beta=c_p-c_q+b^i_\beta+b^t_\beta-b^s_\beta\in F_\beta+B^{<i}_\beta+B^{<i}_\beta-B^{<i}_\beta$, again a contradiction with (b).

Step 3. According to the previous step there exists $\beta>\alpha$ and l such that

$b-c_1=b^i_\beta-b^j_\beta$ where $\max\{i,j\}$ is equal to l;

$b-c_2=b^s_\beta-b^t_\beta$ where $\max\{s,t\}$ is equal to l;

$b-c_3=b^q_\beta-b^r_\beta$ where $\max\{q,r\}$ is equal to l.

In this case we obtain a dichotomy: either among three numbers i,s,q two are equal to l or among j,t,r two are equal to l.

In the fist case we lose no generality assuming that $i=s=l$; in the second, that $j=t=l$.

In the first case we get $F_\alpha\ni c_2-c_1=b^i_\beta-b^s_\beta$, which contradicts (a).

In the second case we get $F_\alpha\ni c_2-c_1=b^t_\beta-b^j_\beta$, which contradicts (a) again.

Therefore, there is no $b\in\bigcap_{i=1}^3(c_i+\mathbb{B}_{>\alpha})\backslash\{c_i\}$ and hence $|\bigcap_{i=1}^3 c_i+\mathbb{B}_{>\alpha}|<\kappa$. □

Acknowledgements. The author expresses her sincere thanks to Taras Banakh and Igor Protasov for valuable and stimulating discussions on the subject of the paper.

References

[BL$_1$] T. Banakh, N. Lyaskovska, *Weakly P-small not P-small subsets in Abelian groups*, Algebra and Discrete Mathematics, No.3 (2006), 29-34.

[BL$_2$] T. Banakh, N. Lyaskovska, *Weakly P-small not P-small subsets in groups*, Intern. J. of Algebra and Computations (to appear).

[CS] J. Convey, N. Sloane, *Sphere packings, lattices, and groups,* Springer, 1993.

[DP] D. Dikranjan, I. Protasov, *Every infinite group can be generated by a P-small subset*, General Applied Topology (to appear); available at http://unicyb.kiev.ua/Site-Eng/admin/download/P-small subset.pdf

[Fu] L.Fuchs, *Infinite abelian groups*, Academic Press, NY, 1970.