SYMMETRIC MONOCHROMATIC SUBSETS IN COLORINGS OF THE LOBACHEVSKY PLANE

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ABSTRACT. We prove that for each partition of the Lobachevsky plane into finitely many Borel pieces one of the cells of the partition contains an unbounded centrally symmetric subset.

It follows from [B1] (see also [BP1, Theorem 1]) that for each partition of the \( n \)-dimensional space \( \mathbb{R}^n \) into \( n \) pieces one of the pieces contains an unbounded centrally symmetric subset. On the other hand, \( \mathbb{R}^n \) admits a partition into \((n+1)\) Borel pieces containing no unbounded centrally symmetric subset. For \( n = 2 \) such a partition is drawn at the picture:

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              B1
                |
                |
                |
                |
                |
B0  B2
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Taking the same partition of the Lobachevsky plane \( H^2 \), we can see that each cell \( B_i \) does contain a unbounded centrally symmetric subset (for such a set just take any hyperbolic line lying in \( B_i \)). We call a subset \( S \) of the hyperbolic plane \( H^2 \) centrally symmetric or else symmetric with respect to a point \( c \in H^2 \) if \( S = f_c(S) \) where \( f_c : H^2 \to H^2 \) is the involutive isometry of \( H^2 \) assigning to each point \( x \in H^2 \) the unique point \( y \in H^2 \) such that \( c \) is the midpoint of the segment \([x, y] \). The map \( f_c \) is called the central symmetry of \( H^2 \) with respect to the point \( c \).

The following theorem shows that the Lobachevsky plane differs dramatically from the Euclidean plane from the Ramsey point of view.

**Theorem 1.** For any partition \( H^2 = B_1 \cup \cdots \cup B_m \) of the Lobachevsky plane into finitely many Borel pieces one of the pieces contains an unbounded centrally symmetric subset.

**Proof.** We shall prove a bit more: given a partition \( H^2 = B_1 \cup \cdots \cup B_m \) of the Lobachevsky plane into \( m \) Borel pieces we shall find \( i \leq m \) and an unbounded subset \( S \subset B_i \) symmetric with respect to some point \( c \) in an arbitrarily small neighborhood of some finite set \( F \subset H^2 \) depending only on \( m \).

To define this set \( F \) it will be convenient to work in the Poincaré model of the Lobachevsky plane \( H^2 \). In this model the hyperbolic plane \( H^2 \) is identified with
the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ on the complex plane and hyperbolic lines are just segments of circles orthogonal to the boundary of $\mathbb{D}$. Let $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ be the hyperbolic plane $\mathbb{D}$ with attached ideal line. For a real number $R > 0$ the set $\mathbb{D}_R = \{z \in \mathbb{C} : |z| \leq 1 - 1/R\}$ can be thought as a hyperbolic disk of increasing radius as $R$ tends to $\infty$.

On the boundary of the unit disk $\mathbb{D}$ consider the $(m+1)$-element set
\[ A = \{z \in \mathbb{C} : z^{m+1} = 1\}. \]
For any two distinct points $x, y \in A$ by $[x|y] \in \mathbb{D}$ we denote the "Euclidean" midpoint of the arc in $\mathbb{D}$ that connects the points $x, y$ and lies on a hyperbolic line in $H^2 = \mathbb{D}$. Then $F = \{[x|y] : x, y \in A, x \neq y\}$ is a finite subset of cardinality $|F| \leq m(m+1)/2$ in the unit disk $\mathbb{D}$.

We claim that for any open neighborhood $W$ of $F$ in $\mathbb{C}$ one of the cells of the partition $H^2 = B_1 \cup \cdots \cup B_m$ contains an unbounded subset symmetric with respect to some point $c \in W$. To derive a contradiction we assume the converse and conclude that for every point $c \in W$ and every $i \leq m$ the set $B_i \cap f_c(B_i)$ is bounded in $H^2$.

For every $n \in \mathbb{N}$ consider the set
\[ C_n = \{c \in W : \bigcup_{i=1}^{m} B_i \cap f_c(B_i) \subset \mathbb{D}_n\}. \]
We claim that $C_n$ is a coanalytic subset of $W$. The latter means that the complement $W \setminus C_n$ is analytic, i.e., is the continuous image of a Polish space. Observe that
\[ W \setminus C_n = \{c \in W : \exists i \leq m \exists x \in \mathbb{D} \setminus \mathbb{D}_n, x \in B_i \text{ and } x \in f_c(B_i)\} = \text{pr}_2(E) \]
where $\text{pr}_2 : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ is the projection on the second factor and
\[ E = \bigcup_{i=1}^{m} \{(x, c) \in \mathbb{D} \times W : x \in \mathbb{D} \setminus \mathbb{D}_n, x \in B_i \text{ and } f_c(x) \in B_i\} \]
is a Borel subset of $\mathbb{D} \times W$. Being a Borel subset of the Polish space $\mathbb{D} \times W$, the space $E$ is analytic and so is its continuous image $\text{pr}_2(E) = W \setminus C_n$. Then $C_n$ is coanalytic and hence has the Baire property [Ke, 21.6], which means that $C_n$ coincides with an open subset $U_n$ of $W$ modulo some meager set. The latter means that the symmetric difference $U_n \triangle C_n$ is meager (that is, of the first Baire category in $W$). Since $C_n \subset C_{n+1}$, we may assume that $U_n \subset U_{n+1}$ for all $n \in \mathbb{N}$. Let $U = \bigcup_{n=1}^{\infty} U_n$ and $M = \bigcup_{n=1}^{\infty} U_n \triangle C_n$.

Taking into account that $W = \bigcup_{n=1}^{\infty} C_n$, we conclude that
\[ W \setminus U = \bigcup_{n=1}^{\infty} C_n \setminus \bigcup_{n=1}^{\infty} U_n \subset \bigcup_{n=1}^{\infty} C_n \setminus U_n \subset \bigcup_{n=1}^{\infty} C_n \triangle U_n = M \]
which implies that the open set $U$ has meager complement and thus is dense in $W$.

We claim that $F \subset h^{-1}(U)$ for some isometry $h$ of the hyperbolic plane $H^2 = \mathbb{D}$. For this consider the natural action
\[ \mu : \text{Iso}(H^2) \times \mathbb{D} \to \mathbb{D}, \quad \mu : (h, x) \mapsto h(x) \]
of the isometry group $\text{Iso}(H^2)$ of the hyperbolic plane $H^2 = \mathbb{D}$. It is easy to see that for every $x \in \mathbb{D}$ the map $\mu_x : \text{Iso}(H^2) \to \mathbb{D}, \mu_x : h \mapsto h(x)$, is continuous and
open (with respect to the compact-open topology on \( \text{Iso}(H^2) \)). It follows that the set 
\[
\bigcap_{x \in F} \mu_x^{-1}(W) = \{ h \in \text{Iso}(H^2) : f(F) \subset W \}
\]
is an open neighborhood of the neutral element of the group \( \text{Iso}(H^2) \).

Taking into account that \( U \) is open and dense in \( W \) and for every \( x \in F \) the map 
\( \mu_x : \text{Iso}(H^2) \to \mathbb{D} \) is open, we conclude that the preimage the set \( \mu_x^{-1}(U) \) is open and dense in \( \mu_x^{-1}(W) \subset \text{Iso}(H^2) \). Then the intersection \( \bigcap_{x \in F} \mu_x^{-1}(U) \), being an open dense subset of \( \bigcap_{x \in F} \mu_x^{-1}(W) \), is not empty and hence contains some isometry \( h \) having the desired property: \( F \subset h^{-1}(U) \). Since \( F \) is finite, there is \( R \in \mathbb{N} \) with \( F \subset h^{-1}(U_R) \). For a complex number \( r \in \mathbb{D} \) consider the set \( rA = \{ rz : z \in A \} \subset \mathbb{D} \) and let
\[
F_r = \{ [x|y] : x, y \in rA, x \neq y \} \subset \mathbb{D},
\]
where \([x|y]\) stands for the midpoint of the hyperbolic segment connecting \( x \) and \( y \) in \( H^2 \). It can be shown that for any distinct points \( x, y \in A \) the “hyperbolic” midpoint \([rx|ry]\) tends to the “Euclidean” midpoint \([x|y]\) as \( r \) tends to \( 1 \). Such a continuity yields a neighborhood \( O_1 \) of \( 1 \) such that \( F_r \subset h^{-1}(U_R) \) for all \( r \in O_1 \cap \mathbb{D} \).

It is clear that for any points \( x, y \in A \) the map
\[
f_{x,y} : \mathbb{D} \to \mathbb{D}, \ f_{x,y} : r \mapsto [rx|ry]
\]
is open and continuous. Consequently, the preimage \( f_{x,y}^{-1}(h^{-1}(M)) \) is a meager subset of \( \mathbb{D} \) and so is the union \( M' = \bigcup_{x, y \in A} f_{x,y}^{-1}(h^{-1}(M)) \). So, we can find a non-zero point \( r \in O_1 \setminus M' \) so close to \( 1 \) that the set \( rA \) is disjoint with the hyperbolic disk \( h^{-1}(D_R) \). For this point \( r \) we shall get \( F_r \cap h^{-1}(M) = \emptyset \).

The set \( rA \) consists of \( m + 1 \) points. Consequently, some cell \( h^{-1}(B_i) \) of the partition \( \mathbb{D} = h^{-1}(B_1) \cup \cdots \cup h^{-1}(B_m) \) contains two distinct points \( rx, ry \) of \( rA \). Those points are symmetric with respect to the point
\[
[rx|ry] \in F_r \subset h^{-1}(U_R) \setminus h^{-1}(M).
\]
Then the images \( a = h(rx) \) and \( b = h(ry) \) belong to \( B_i \) and are symmetric with respect to the point \( c = h([rx|ry]) \in U_R \setminus M \subset C_R \). It follows from the definition of \( C_R \) that \( \{ a, b \} \subset B_i \cap f_{c}(B_i) \subset \mathbb{D} \), which is not the case because \( rx, ry \notin h^{-1}(D_R) \).

We do not know if Theorem 1 is true for any finite (not necessarily Borel) partition of the Lobachevsky plane \( H^2 \). For partitions of \( H^2 \) into two pieces the Borel assumption is superfluous.

**Theorem 2.** There is a subset \( T \subset H^2 \) of cardinality \( |T| = 3 \) such that for any partition \( H^2 = A_1 \cup A_2 \) of \( H^2 \) into two pieces either \( A_1 \) or \( A_2 \) contains an unbounded subset, symmetric with respect to some point \( c \in T \).

**Proof.** Lemma 1 below allows us to find an equilateral triangle \( \triangle c_0c_1c_2 \) on the Lobachevsky plane \( H^2 \) such that the composition \( f_{c_2} \circ f_{c_1} \circ f_{c_0} \) of the symmetries with respect to the points \( c_0, c_1, c_2 \) coincides with the rotation on the angle \( 2\pi/3 \) around some point \( o \in H^2 \). Consequently \( (f_{c_2} \circ f_{c_1} \circ f_{c_0})^3 \) is the identity isometry of \( H^2 \).

We claim that for any partition \( H^2 = A_1 \cup A_2 \) of the Lobachevsky plane into two pieces one of the pieces contains an unbounded subset symmetric with respect
to some point in the triangle \( T = \{c_0, c_1, c_2\} \). Assuming the converse, we conclude that the set

\[
B = \bigcup_{c \in T} \bigcup_{i=1}^{2} A_i \cap f_c(A_i)
\]

is bounded. It follows that two points \( x, y \in H^2 \setminus B \), symmetric with respect to a center \( c \in T \) cannot belong to the same cell \( A_i \) of the partition.

Given a point \( x_0 \in H^2 \) consider the sequence of points \( x_1, \ldots, x_9 \) defined by the recursive formula:

\[
x_{i+1} = f_{c_i \mod 3}(x_i)
\]

It follows that \( x_9 = (f_{c_2} \circ f_{c_1} \circ f_{c_0})^3(x_0) = x_0 \).

Taking \( x_0 \) sufficiently far from the center \( o \) of rotation we can guarantee that none of the points \( x_0, \ldots, x_9 \) belongs to \( B \).

The point \( x_0 \) belongs either to \( A_1 \) or to \( A_2 \). We lose no generality assuming that \( x_0 \notin A_2 \). Since the points \( x_0, x_1 \notin B \) are symmetric with respect to \( c_0 \) and \( x_0 \notin A_2 \), we get that \( x_1 \in H^2 \setminus A_2 = A_1 \). By the same reason \( x_1, x_2 \) cannot simultaneously belong to \( A_1 \) and hence \( x_2 \notin A_2 \). Continuing in this fashion we conclude that \( x_i \) belongs to \( A_1 \) for odd \( i \) and to \( A_2 \) for even \( i \). In particular, \( x_9 \in A_1 \), which is not possible because \( x_9 = x_0 \notin A_2 \).

\( \square \)

**Lemma 1.** There is an equilateral triangle \( \triangle ABC \) on the Lobachevsky plane such that the composition \( f_C \circ f_B \circ f_A \) of the symmetries with respect to the points \( A, B, C \) coincides with the rotation on the angle \( 2\pi/3 \) around some point \( O \).

**Proof.** For a positive real number \( t \) consider an equilateral triangle \( \triangle ABC \) with side \( t \) the on the Lobachevsky plane. Let \( M \) be the midpoint of the side \( AB \) and \( l \) be the line through \( C \) that is orthogonal to the line \( CM \). Consider also the line \( p \) that is orthogonal to the line \( AB \) and passes through the point \( P \) such that \( A \) is the midpoint between \( P \) and \( M \). Observe that \( |PM| = |AB| = t \) and for sufficiently small \( t \) the lines \( p \) and \( l \) intersect at some point \( O \).

![Diagram](image)

It is easy to see that the composition \( f_B \circ f_A \) is the shift along the line \( AB \) on the distance \( 2t \) and hence the image \( f_B \circ f_A(O) \) of the point \( O \) is the point symmetric to \( O \) with respect to the point \( C \). Consequently, \( f_C \circ f_B \circ f_A(O) = O \), which means...
that the isometry $f_C \circ f_B \circ f_A$ is a rotation of the Lobachevsky plane around the point $O$ on some angle $\varphi_t$.

To estimate this angle, consider the point $X$ such that $P$ is the midpoint between $X$ and $M$. Then $|XM| = 2t$ and consequently, $f_B \circ f_A(X) = M$ while $X' = f_C \circ f_B \circ f_A = f_C(M)$ is the point on the line $CM$ such that $C$ is the midpoint between $X'$ and $M$. It follows that $|X'X| = |XM| + |MX'| < 2t + 2t = 4t$.

Observe that for small $t$ the point $X'$ is near to the point, symmetric to $X$ with respect to $O$, which means that the angle $\varphi_t = \angle XOX'$ is close to $\pi$ for $t$ close to zero. On the other hand, for very large $t$ the lines $p$ and $l$ on the Lobachevsky plane do not intersect. So we can consider the smallest upper bound $t_0$ of numbers $t$ for which the lines $l$ and $p$ meet. For values $t < t_0$ near to $t_0$ the point $O$ tends to infinity as $t$ tends to $t_0$. Since the length of the side $XX'$ of the triangle $\triangle XOX'$ is bounded by $4t_0$ the angle $\varphi_t = \angle XOX'$ tends to zero as $O$ tends to infinity. Since the angle $\varphi_t$ depends continuous on $t$ and decreases from $\pi$ to zero as $t$ increases from zero to $t_0$, there is a value $t$ such that $\varphi_t = 2\pi/3$. For such $t$ the composition $f_C \circ f_B \circ f_A$ is the rotation around $O$ on the angle $2\pi/3$. 

\section*{Some Comments and Open Problems}

In contrast to Theorem 1, Theorem 2 is true for the Euclidean plane $E^2$ even in a stronger form: for any subset $C \subset E^2$ not lying on a line and any partition $E^2 = A_1 \cup A_2$ one of the cells of the partition contains an unbounded subset symmetric with respect to some center $c \in C$, see [B2].

Having in mind this result let us call a subset $C$ of a Lobachevsky or Euclidean space $X$ \textit{central for (Borel) $k$-partitions} if for any partition $X = A_1 \cup \cdots \cup A_k$ of $X$ into $k$ (Borel) pieces one of the pieces contains an unbounded monochromatic subset symmetric with respect to some point $c \in C$. By $c_k(X)$ (resp. $c_k^B(X)$) we shall denote the smallest size of a subset $C \subset X$, central for (Borel) $k$-partitions of $X$. If no such a set $C$ exists, then we put $c_k(X) = \infty$ (resp. $c_k^B(X) = \infty$) where $\infty$ is assumed to be greater than any cardinal number. It follows from the definition that $c_k^B(X) \leq c_k(X)$.

We have a lot of information on the numbers $c_k^B(E^n)$ and $c_k(E^n)$ for Euclidean spaces $E^n$, see [B2]. In particular, we know that

1. $c_2(E^n) = c_2^B(E^n) = 3$ for all $n \geq 2$;
2. $c_3(E^3) = c_3^B(E^3) = 6$;
3. $12 \leq c_4^B(E^4) \leq c_4(E^4) \leq 14$;
4. $n(n + 1)/2 \leq c_n^B(E^n) \leq c_n(E^n) \leq 2^n - 2$ for every $n \geq 3$.

Much less is known on the numbers $c_k^B(H^n)$ and $c_k(H^n)$ in the hyperbolic case. Theorem 2 yields the upper bound $c_2(H^2) \leq 3$. In fact, 3 is the exact value of $c_2(H^n)$ for all $n \geq 2$.

\textbf{Proposition 1.} $c_k^B(H^n) = c_k(H^n) = 3$ for all $n \geq 2$.

\textit{Proof.} The upper bound $c_2(H^n) \leq c_2(H^2) \leq 3$ follows from Theorem 2. The lower bound $3 \leq c_2^B(H^n)$ will follow as soon as for any two points $c_1, c_2 \in H^n$ we construct a partition $H^n = A_1 \cup A_2$ in two Borel pieces containing no unbounded set, symmetric with respect to a point $c_i$. To construct such a partition, consider the line $l$ containing the points $c_1, c_2$ and decompose $l$ into two half-lines $l = l_1 \cup l_2$.

Next, let $H$ be an $(n - 1)$-hyperplane in $H^n$, orthogonal to the line $l$. Let $S$ be the unit sphere in $H$ centered at the intersection point of $l$ and $H$. Let $S = B_1 \cup B_2$
be a partition of $S$ into two Borel pieces such that no antipodal points of $S$ lie in the same cell of the partition. For each point $x \in H^n \setminus l$ consider the hyperbolic plane $P_x$ containing the points $x, c_1, c_2$. The complement $P_x \setminus l$ decomposes into two half-planes $P^+_x \cup P^-_x$ where $P^+_x$ is the half-plane containing the point $x$. The plane $P_x$ intersects the hyperplane $H$ by a hyperbolic line containing two points of the sphere $S$. Finally put

$$A_i = l_i \cup \{x \in H^2 \setminus l : P^+_x \cap B_i \neq \emptyset\}$$

for $i \in \{1, 2\}$. It is easy to check that $A_1 \cup A_2 = H^n$ is the desired partition of the hyperbolic space into two Borel pieces none of which contains an unbounded subset symmetric with respect to one of the points $c_1, c_2$.

The preceding proposition implies that the cardinal numbers $c_2(H^n)$ are finite.

**Problem 1.** For which numbers $k, n$ are the cardinal numbers $c_k(H^n)$ and $c_k^B(H^n)$ finite? Is it true for all $k \leq n$?

Except for the equality $c_2(E^n) = 3$, we have no information on the numbers $c_k(E^n)$ with $k < n$.

**Problem 2.** Calculate (or at least evaluate) the numbers $c_k(E^n)$ and $c_k^B(H^n)$ for $2 < k < n$.

In all the cases where we know the exact values of the numbers $c_k(E^n)$ and $c_k^B(H^n)$ we see that those numbers are equal.

**Problem 3.** Are the numbers $c_k(E^n)$ and $c_k^B(E^n)$ (resp. $c_k(H^n)$ and $c_k^B(H^n)$) equal for all $k, n$?

Having in mind that each subset not lying on a line is central for 2-partitions of the Euclidean plane, we may ask about the same property of the Lobachevsky plane.

**Problem 4.** Is any subset $C \subset H^2$ not lying on a line central for (Borel) 2-partitions of the Lobachevsky plane $H^2$?

Finally, let us ask about the numbers $c_k^B(H^2)$ and $c_k(H^2)$. Observe that Theorem 1 guarantees that $c_k^B(H^2) \leq \kappa$ for all $k \in \mathbb{N}$. Inspecting the proof we can see that this upper bound can be improved to $c_k^B(H^2) \leq \text{non}(\mathcal{M})$ where $\text{non}(\mathcal{M})$ is the smallest cardinality of a non-meager subset of the real line. It is clear that $\kappa_1 \leq \text{non}(\mathcal{M}) \leq \kappa$. The exact location of the cardinal $\text{non}(\mathcal{M})$ on the interval $[\kappa_1, \kappa]$ depends on axioms of Set Theory, see [Bl]. In particular, the inequality $\kappa_1 = \text{non}(\mathcal{M}) < \kappa$ is consistent with ZFC.

**Problem 5.** Is the inequality $c_k^B(H^2) \leq \kappa_1$ provable in ZFC? Are the cardinals $c_k^B(H^2)$ countable? finite?

The latter problem asks if $H^2$ contains a countable (or finite) central set for Borel $k$-partitions of the Lobachevsky plane. Inspecting the proof of Theorem 1 we can see that it gives an “approximate” answer to this problem:

**Proposition 2.** For any $k \in \mathbb{N}$ there is a finite subset $C \subset H^2$ of cardinality $|C| \leq k(k+1)/2$ such that for any partition $H^2 = B_1 \cup \cdots \cup B_k$ of $H^2$ into $k$ Borel pieces and for any open neighborhood $O(C) \subset H^2$ of $C$ one of the pieces $B_i$ contains an unbounded subset $S \subset B_i$ symmetric with respect to some point $c \in O(C)$. 

Remark 1. For further results and open problems related to symmetry and colorings see the surveys [BP$_2$], [BVV] and the list of problems [BBGRZ, §4].

REFERENCES


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