

CHARACTERIZING METRIC SPACES WHOSE HYPERSPACES ARE ABSOLUTE NEIGHBORHOOD RETRACTS

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ABSTRACT. We characterize metric spaces X whose hyperspaces 2^X or $\text{Bd}(X)$ of non-empty closed (bounded) subsets, endowed with the Hausdorff metric, are absolute [neighborhood] retracts.

One of the principal results linking Theory of Hyperspaces with Theory of Retracts is Wojdysławski Theorem [Wo] asserting that the hyperspace 2^X of a compact metric space X is an absolute retract if and only if X is a Peano continuum that is a continuous image of the interval $[0, 1]$. Among many characterizations of Peano continua let us recall the Hahn-Mazurkiewicz-Sierpiński Theorem (see [Ku, §50]) asserting that a connected metrizable compact space is a Peano continuum if and only if it is locally connected and the Bing convexification theorem [Bi], [Mo] characterizing Peano continua as compacta admitting a convex metric. We recall that a metric d on X is *convex* (resp. *almost convex*) if for any $x, y \in X$ and positive reals s, t with $d(x, y) \leq s + t$ (resp. $d(x, y) < s + t$) there is $z \in X$ such that $d(x, z) \leq s$ and $d(z, y) \leq t$. Each almost convex metric on a compact space is convex. On the other hand, the standard metric on the space of rational numbers is almost convex but fails to be convex.

Combining Bing's and Wojdysławski's theorems we conclude that for a compact space X endowed with a convex metric d the hyperspace 2^X of non-empty closed subsets of X is an absolute retract. Generalizing this result to non-compact spaces, C.Costantini and W.Kubiś [CK] proved that for an almost convex bounded metric space X the hyperspace 2^X of all non-empty closed subsets of X , endowed with the Hausdorff metric, is an absolute neighborhood retract. In its turn M.Kurihara, K.Sakai and M.Yaguchi [KSY] showed that the almost convexity of X in the above result can be replaced by the so-called uniform local C^* -connectedness (in the present paper this property is called the uniform local chain equi-connectedness).

In this paper we show that the latter result of [KSY] can be reversed. More precisely, the hyperspace 2^X of all non-empty closed subsets of a metric space X is an ANR (and AR) if and only if X is uniformly locally chain equi-connected (and chain equi-connected). It is interesting to compare this result with characterization of the ANR-property in other subspaces of 2^X : for a (connected) metric space X the hyperspace $\mathcal{F}(X)$ of all non-empty finite subsets of X is an ANR (an AR) if and only if X is locally path connected [CN] while the hyperspace $\mathcal{K}(X)$ of all non-empty compact subsets of X is an ANR (an AR) if and only if X is locally continuum connected, see [Cu].

Now it is time to give precise definitions. For a bounded metric space (X, d) let 2^X denote the hyperspace of all non-empty closed subsets of X , endowed with the Hausdorff metric

$$d_H(A, B) = \max\{\sup_{x \in B} d(x, A), \sup_{x \in A} d(x, B)\}.$$

For an unbounded metric space (X, d) the Hausdorff metric can attain infinite values but still determined a topology on 2^X , called the uniform topology. This topology depends not on a particular metric d on X but on the uniformity generated by that metric, see [En, Ch.8] for the theory of uniform spaces.

Each uniformity \mathcal{U} generating the topology of a space X induces the *Hausdorff uniformity* $2^{\mathcal{U}}$ on the hyperspace 2^X of all non-empty closed subsets of X . This uniformity $2^{\mathcal{U}}$ is generated by

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the base consisting of the entourages

$$2^U = \{(A, A') \in 2^X \times 2^X : A \subset B(A', U), A' \subset B(A, U)\}, \quad U \in \mathcal{U}.$$

Here, as expected, $B(A, U) = \bigcup_{a \in A} B(a, U)$ where $B(a, U) = \{x \in X : (x, a) \in U\}$ is the U -ball centered at $a \in X$, see [En, 8.5.16]. The topology on 2^X induced by the Hausdorff uniformity $2^{\mathcal{U}}$ will be called the *uniform topology* on 2^X . Talking about topological properties of 2^X we shall always refer to this (uniform) topology.

If the uniformity \mathcal{U} on X is generated by a metric d , then the uniformity of the subspace $\text{Bd}(X) \subset 2^X$ consisting of all bounded non-empty closed subsets of X is generated by the Hausdorff metric

$$d_H(A, B) = \max\{\sup_{x \in B} d(x, A), \sup_{x \in A} d(x, B)\},$$

[En, 8.5.16.b]. For arbitrary (non-necessarily bounded) closed subsets A, B of the metric space (X, d) the Hausdorff distance $d_H(A, B)$ can attain infinite values, but letting $\rho_H = \min\{1, d_H\}$ we get a metric inducing the uniformity $2^{\mathcal{U}}$ on the whole hyperspace 2^X .

It should be mentioned that the uniform topology on 2^X coincides with the Vietoris topology if and only if X is totally bounded. On the other hand, these two topologies always coincide on the subspace $\mathcal{K}(X) \subset 2^X$ of all non-empty compact subsets of X , see [En, 8.5.16(c)].

The uniform as well as the Vietoris topologies on 2^X were actively studied by general topologists, see [En], [Be], [IN]. In spite of their coincidence on the hyperspace $\mathcal{K}(X)$, for non-compact X the uniform and Vietoris topologies on 2^X differ substantially by their connectedness properties. In particular, for a connected uniform space X the hyperspace 2^X endowed with the Vietoris topology is connected but rarely is locally connected. On the other hand, the hyperspace 2^X endowed with the uniform topology rarely is connected but often is locally connected. A typical example of this phenomenon is the hyperspace $2^{\mathbb{R}}$ of the real line \mathbb{R} . Endowed with the Vietoris topology $2^{\mathbb{R}}$ is connected but fails to be locally connected. In contrast, endowed with the uniform topology the hyperspace $2^{\mathbb{R}}$ is locally connected but fails to be connected.

Striving to characterize metrizable uniform spaces X whose hyperspace 2^X is an ANR we invent that the ANR-property in hyperspaces is equivalent to a wide spectrum of local properties having topological, uniform, metric or extension nature.

We start with two local properties having topological nature. We recall that a topological space X is

- *locally path connected* [briefly (lpc)] if for each $x_0 \in X$ and a neighborhood $U \subset X$ of x_0 there is a neighborhood $V \subset U$ of x_0 such that each point $x \in V$ can be linked with x_0 by a continuous path $f : [0, 1] \rightarrow U$ with $f(0) = x_0, f(1) = x$;
- *locally connected* [briefly (lc)] if for each $x_0 \in X$ and a neighborhood $U \subset X$ of x_0 there is a connected subset $C \subset U$, containing a neighborhood of x_0 .

Next, we consider some properties having uniform nature. In some cases defining certain property we shall simultaneously (in parentheses) define its relative versions. For a point x in a metric space (X, d) by $B(x, r) = \{y \in X : d(x, y) < r\}$ we denote the open r -ball around x . We shall say that a metric space (X, d) is

- *uniformly locally compact* if there is $\varepsilon > 0$ such that each closed ε -ball $\bar{B}(x, \varepsilon) = \{y \in X : d(x, y) \leq \varepsilon\}$ is compact;
- *chain connected* if for any points $x, y \in X$ and any $\eta > 0$ there is a sequence $x = x_0, x_1, \dots, x_l = y$ of points of X such that $d(x_i, x_{i-1}) < \eta$ for all $i \leq l$; such a sequence $x = x_0, x_1, \dots, x_l = y$ is called an η -chain linking x and y and l is the *length* of this chain;
- *chain equi-connected* if for any $\eta > 0$ there is a number $l \in \mathbb{N}$ such that any points $x, y \in X$ can be connected by an η -chain of length $\leq l$;
- *chain connected im kleinen* if there is $\varepsilon > 0$ such that any points $x, y \in X$ with $d(x, y) < \varepsilon$ can be linked by an η -chain for any $\eta > 0$;
- *locally chain connected* [briefly (lcc)] (at a subset $X_0 \subset X$) if for each point x_0 in X (in X_0) and $\varepsilon > 0$ there is $\delta > 0$ such that for each $\eta > 0$ and x in $B(x_0, \delta)$ (in $X_0 \cap B(x_0, \delta)$) there is an η -chain $x_0, x_1, \dots, x_l = x$ of diameter $< \varepsilon$ linking the points x_0 and x ;

- *uniformly locally chain connected* [briefly (ulcc)] if $\forall \varepsilon > 0 \exists \delta > 0 \forall \eta > 0$ such that any points $x, y \in X$ with $d(x, y) < \delta$ can be connected by an η -chain of diameter $< \varepsilon$;
- *uniformly locally chain equi-connected* [briefly (ulcec)] (at a subset $X_0 \subset X$) if $\forall \varepsilon > 0 \exists \delta > 0 \forall \eta > 0 \exists l \in \mathbb{N}$ such that any points x, y in X (in X_0) with $d(x, y) < \delta$ can be connected by an η -chain of diameter $< \varepsilon$ and length $\leq l$;
- *uniformly locally path connected* [briefly (ulpc)] if for any $\varepsilon > 0$ there is $\delta > 0$ such that any points x, y with $d(x, y) < \delta$ can be connected by a continuous path $f : [0, 1] \rightarrow X$ of diameter $< \varepsilon$ such that $f(0) = x$ and $f(1) = y$.
- *locally trace connected* [briefly (ltc)] if for each point $x_0 \in X$ and $\varepsilon > 0$ there is $\delta > 0$ such that for any point $x \in X$ with $d(x, x_0) < \delta$ and a countable dense subset $Q \supset \{0, 1\}$ of $[0, 1]$ there is a uniformly continuous map $f : Q \rightarrow B(x_0, \varepsilon)$ such that $f(0) = x_0$ and $f(1) = x$; such a uniformly continuous map $f : Q \rightarrow X$ is called a *trace* linking x_0 and x ;
- *uniformly locally trace connected* [briefly (ultc)] if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any countable dense set $Q \supset \{0, 1\}$ of $[0, 1]$ any points x, y with $d(x, y) < \delta$ can be connected by a trace $f : Q \rightarrow X$ with $\text{diam}f(Q) < \varepsilon$.

Next, let us consider connectedness properties having metric nature. By the *continuity modulus* of a function $f : X \rightarrow Y$ between metric spaces (X, d) , (Y, ρ) we understand the non-decreasing function $\omega_f : (0, \infty) \rightarrow (0, \infty]$ defined by

$$\omega_f(t) = \sup\{\rho(f(x), f(x')) : x, x' \in X, d(x, x') \leq t\} \text{ for } t \in [0, \infty).$$

If the metric space (X, d) is almost convex, then the continuity modulus ω_f is *subadditive* in the sense that $\omega_f(s+t) \leq \omega_f(s) + \omega_f(t)$. Let us remark that the map f is *uniformly continuous* if and only if $\lim_{t \rightarrow +0} \omega_f(t) = 0$.

In the sequel by a *continuity modulus* we shall understand an arbitrary non-decreasing function $\omega : (0, \infty) \rightarrow (0, \infty)$ with $\lim_{t \rightarrow +0} \omega(t) = 0$.

We shall say that a metric space (X, d) is

- *uniformly locally path equi-connected* [briefly (ulpec)] if there are $\varepsilon_0 > 0$ and a continuity modulus ω such that any two points $x, y \in X$ with $d(x, y) < \varepsilon_0$ can be connected by a path $f : [0, d(x, y)] \rightarrow X$ such that $f(0) = x$, $f(d(x, y)) = y$, and $\omega_f \leq \omega$;
- *uniformly locally trace equi-connected* [briefly (ultec)] (at a subset $X_0 \subset X$) if there are $\varepsilon_0 > 0$ and a continuity modulus ω such that for any two points x, y in X (in X_0) with $d(x, y) < \varepsilon_0$ and any countable dense subset $Q \supset \{0, d(x, y)\}$ of $[0, d(x, y)]$ there is a uniformly continuous function $f : Q \rightarrow X$ such that $f(0) = x$, $f(d(x, y)) = y$, and $\omega_f \leq \omega$;
- *ω -convex*, where ω is a continuity modulus, if for any points $x, y \in X$ and positive reals s, t with $d(x, y) < \omega(s+t)$ there is a point $z \in X$ such that $d(x, z) < \omega(s)$ and $d(z, y) < \omega(t)$.

Let us remark that a metric space X is almost convex if and only if it is ω -convex for a linear continuity modulus $\omega(t) = at$.

By the *chain connected component* of a point x_0 in a metric space (X, d) we understand the set $C(x_0) = \{x \in X : \text{for any } \eta > 0 \text{ the points } x \text{ and } x_0 \text{ can be linked by an } \eta\text{-chain in } X\}$. We shall say that a uniform space X is

- *weakly convexifiable* [briefly (wcx)] if X is chain connected im kleinen and there is a concave continuity modulus α such that for each concave continuity modulus $\omega \geq \alpha$ the uniformity of X is generated by a metric which is ω -convex on each chain connected component of X .

We recall that a function $f : [0, \infty) \rightarrow [0, \infty)$ is *concave* if

$$f(ta + (1-t)b) \geq tf(a) + (1-t)f(b)$$

for any $a, b \in [0, \infty)$ and $t \in [0, 1]$.

Finally we define another three important local extension properties of metric spaces which in case of hyperspaces are near to the local connectedness properties discussed above. By a *uniform neighborhood* of a set X in a metric space (M, d) we understand any subset $U \subset M$

containing the ε -neighborhood $B(X, \varepsilon) = \{x \in M : d(x, X) < \varepsilon\}$ of X for some $\varepsilon > 0$. We shall say that a metric space (X, d) is

- an *absolute neighborhood retract* [briefly ANR] if for each metric space M containing X isometrically there is a retraction $r : U \rightarrow X$ defined on some open neighborhood U of X in M ;
- an *absolute neighborhood uniform retract* [briefly ANUR] if for each metric space M containing X isometrically there is a uniformly continuous retraction $r : U \rightarrow X$ defined on some uniform neighborhood U of X in M ;
- a *uniform absolute neighborhood retract* [briefly uANR] if for each metric space M containing X isometrically there is a retraction $r : U \rightarrow X$ defined on some uniform neighborhood U of X in M and such that r is uniformly continuous at X in the sense that for any $\varepsilon > 0$ there is $\delta > 0$ such that $d(f(y), x) < \varepsilon$ for any $x \in X$ and $y \in U$ with $d(x, y) < \delta$.

Replacing in the above definition the neighborhood U of X by the whole M we will get the definitions of an *absolute retract*, *absolute uniform retract*, and *uniform absolute retract* [briefly, (AR), (AUR), and (uAR)].

It is interesting to note that the theories of ANUR's and uANR's have different historical origins. ANUR's arose in Nonlinear Functional Analysis [BL] and were studied mainly by analysts while uANR's first appeared in the topological paper of Michael [Mi] and were studied mainly by topologists, see [Sa].

We shall show that for each metric space X the properties above-defined relate as follows (the vertical arrows \uparrow correspond to the implications holding under the additional assumption of the *completeness* of X):

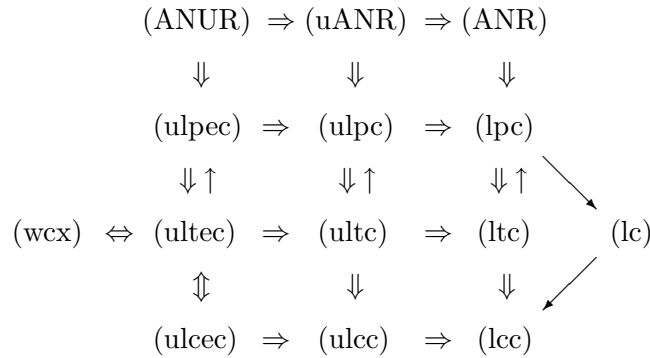


Diagram 1

Except for the equivalences $(\text{ulcec}) \Leftrightarrow (\text{ultec}) \Leftrightarrow (\text{wcx})$ (proven in Section 1) all the implications of Diagram 1 are rather trivial (to see that the extension properties from the first row of Diagram 1 imply the path-connectedness properties from the second row, apply Eels-Arens-Wojdysławski Theorem (see [BP, II.§1]) asserting that each metric space is isometric to a closed subset of a linear normed space).

It turns out that for the hyperspace 2^X of a metric space X all these 14 properties (with possible exception of ANUR) are equivalent.

Besides the hyperspace 2^X of a uniform space X we shall be interested in some its subspaces:

- $\mathcal{D}(X) = \{D \in 2^X : D \text{ is discrete}\}$
- $u\mathcal{D}(X) = \{D \in 2^X : D \text{ is uniformly discrete in } X\}$ and
- $\mathcal{D}_0(X) = \{D \in \mathcal{D}(X) : |D| \leq \aleph_0 \text{ and } D \text{ is totally bounded or uniformly discrete in } X\}$.

We call a subset D of a metric space (X, d) *uniformly discrete* if there is $\varepsilon > 0$ such that $|F \cap B(x, \varepsilon)| \leq 1$ for each point $x \in X$. It is easy to see that a subset $F \subset X$ is uniformly discrete in X if and only if F is ε -separated for some $\varepsilon > 0$ in the sense that $d(x, y) \geq \varepsilon$ for all distinct $x, y \in F$. Using the Zorn Lemma it is easy to show that the set $u\mathcal{D}(X)$ is dense in 2^X . Note also that for a complete metric space X the set $\mathcal{D}_0(X)$ coincides with the collection of

all at most countable uniformly discrete subsets of X (because closed discrete totally bounded subsets in complete uniform spaces are finite).

The main result of this paper is the following characterizing

Theorem 1. *For a metric space (X, d) and its hyperspace 2^X endowed with the “Hausdorff” $\min\{1, d_H\}$ the following conditions are equivalent:*

- (1) 2^X is an ANR;
- (2) 2^X is locally connected;
- (3) 2^X is a uniform ANR;
- (4) 2^X is uniformly locally path equi-connected;
- (5) 2^X is locally chain connected;
- (6) 2^X is locally chain connected at $\mathcal{D}_0(X)$;
- (7) X is uniformly locally chain equi-connected;
- (8) X is uniformly locally trace equi-connected;
- (9) X is weakly convexifiable.

Moreover, if the space X is complete, then the conditions (1)–(9) are equivalent to

- (10) X is uniformly locally path equi-connected;

If X is uniformly locally compact, then the conditions (1)–(10) are equivalent to

- (11) Each space $\mathcal{H} \subset 2^X$ containing $\mathcal{D}(X)$ is a uniform ANR;

If X is chain equi-connected, then the conditions (1)–(9) are equivalent to

- (12) 2^X is an absolute retract;
- (13) 2^X is a uniform absolute retracts.

The implication (7) \Rightarrow (3) was first proved in [KSY] while some particular cases of (10) \Rightarrow (11) were proved by M.Kurihara, see [KSY]. By Proposition 4.6 of [CK], the hyperspace 2^X of a metric space X is chain connected if and only if X is chain equi-connected. Combining this result with Theorem 1 we get characterizing

Theorem 2. *The hyperspace 2^X of a metric space X is an absolute retract if and only if 2^X is a uniform absolute retract if and only if X is chain equi-connected and uniformly locally chain equi-connected.*

Some implications of Theorem 1 hold in a more general setting. For a closed subset F of a metric space X let $\mathcal{D}_0(F) = \{D \in \mathcal{D}_0(X) : D \subset F\}$.

Theorem 3. *For a metric space X and an open subspace $\mathcal{H} \subset 2^X$ such that $\mathcal{D}_0(F) \subset \mathcal{H}$ for each $F \in \mathcal{H}$ the following conditions are equivalent:*

- (1) \mathcal{H} is an ANR;
- (2) \mathcal{H} is locally path connected;
- (3) \mathcal{H} is locally chain connected;
- (4) 2^X is locally chain connected at $\mathcal{D}_0(F)$ for each $F \in \mathcal{H}$;
- (5) X is uniformly locally chain equi-connected at each $F \in \mathcal{H}$;
- (6) X is uniformly locally trace equi-connected at each $F \in \mathcal{H}$;

Applying Theorem 3 to the hyperspace $\mathcal{H} = \text{Bd}(X) \subset 2^X$ of all closed bounded subsets of a metric space X we get

Corollary 1. *For the hyperspace $\text{Bd}(X) \subset 2^X$ of all non-empty closed bounded subsets of a metric space X the following conditions are equivalent:*

- (1) $\text{Bd}(X)$ is an ANR;
- (2) $\text{Bd}(X)$ is locally path connected;
- (3) $\text{Bd}(X)$ is locally chain connected;
- (4) $\text{Bd}(X)$ is locally chain connected at $\text{Bd}(X) \cap \mathcal{D}_0(X)$;
- (5) X is uniformly locally chain equi-connected at each bounded subset of X ;
- (6) X is uniformly locally trace equi-connected at each bounded subset of X ;

Theorem 1 allows us to characterize the AR-property in the hyperspaces $\text{Bd}(X)$.

Theorem 4. *The hyperspace $\text{Bd}(X)$ of a metric space X is an absolute retract if and only if*

- (i) *each bounded subset of X lies in a bounded chain equi-connected subspace of X and*
- (ii) *X is uniformly locally chain equi-connected at each bounded subset of X .*

Problem 1. *Characterize metric spaces whose hyperspace is an absolute (neighborhood) uniform retract.*

The above characterizations imply several unexpected corollaries.

Corollary 2. *Let X be a dense subset of a metric space M .*

- (1) *The hyperspace 2^X is an absolute (neighborhood) retract if and only if so is the hyperspace 2^M .*
- (2) *The hyperspace $\text{Bd}(X)$ is an absolute (neighborhood) retract if and only if so is the hyperspace $\text{Bd}(M)$.*

Another unexpected corollary is an amusing characterization of normable spaces. We recall that a linear topological space X is *normable* if its topology is determined by a norm.

Corollary 3. *A metrizable locally convex space X is normable if and only if its hyperspace 2^X is an ANR.*

Proof. Each normed space X carries a convex metric and consequently is uniformly locally chain equi-connected. Applying Theorem 1, we conclude that the hyperspace 2^X is an ANR.

Assume conversely that the hyperspace 2^X of a metrizable locally convex space X is an ANR. Then Theorem 1 implies that X is uniformly locally chain equi-connected. This means that there is a convex neighborhood U of the origin of X such that for any convex neighborhood $W \subset X$ of the origin there is $l \in \mathbb{N}$ such that for any $x \in U$ there is a chain $0 = x_0, \dots, x_l = x$ with $x_i - x_{i-1} \in W$ for all $i \leq l$. This implies that $\underbrace{W + \dots + W}_l = lW \supset U$, which means that

the set U is bounded in X . Then X contains a bounded convex neighborhood U of the origin and hence X is normable, see [Sch, II.2.1]. \square

This corollary implies that the hyperspace $2^{\mathbb{R}^\omega}$ of \mathbb{R}^ω , the countable product of lines, fails to be an ANR (in spite of the fact that \mathbb{R}^ω is an absolute uniform retract). The first example of a metric absolute retract X whose hyperspace 2^X fails to be an ANR was constructed in [KSY].

Problem 2. *Characterize metric linear spaces whose hyperspaces are ANR's.*

Finally let us pose an intriguing open problem related to the Bing convexification Theorem [Bi] and the implication (2) \Rightarrow (9) of Theorem 1. According to this implication, the uniformity of any chain connected metric space X with locally connected hyperspace 2^X is generated by a ω -convex metric for some concave continuity modulus ω . Let us remark that a metric d is almost convex if and only if it is ω -convex for a linear continuity modulus $\omega(t) = at$.

Problem 3. *Assume that the hyperspace 2^X of some chain connected metric space X is an ANR. Is the uniformity of X generated by an almost convex metric?*

The answer to this problem is affirmative if X is totally bounded.

Theorem 5. *The hyperspace 2^X of a totally bounded metric space is an absolute retract if and only if the uniformity of X is generated by an almost convex metric.*

Proof. The “if” part of this theorem was proved in [CK] (and can be derived from Theorem 2). To prove the “only if” part, assume that 2^X is an AR. Since X is totally bounded, the completion Y of X is compact. Since X is dense in Y , Corollary 2 implies that the hyperspace 2^Y is an AR. Combining Wojdysławski's and Bing's Theorems we conclude that the uniformity of Y is generated by a convex metric d . Then d restricted to X is almost convex and generates the uniformity of X . \square

Now let us pass to the proofs of our results.

1. PROOFS OF THE IMPLICATIONS FROM DIAGRAM 1

In this section we shall prove the non-trivial equivalences $(\text{ulcec}) \Leftrightarrow (\text{ultec}) \Leftrightarrow (\text{wcx})$ from Diagram 1.

Lemma 1. *If a metric space (X, d) is uniformly locally trace equi-connected at a subset $X_0 \subset X$, then X is uniformly locally chain equi-connected at X_0 .*

Proof. Assuming that X is uniformly locally trace equi-connected at X_0 , find $\varepsilon_0 > 0$ and a continuity modulus ω such that any two points $x, y \in X_0$ with $d(x, y) < \varepsilon_0$ can be linked by a trace $f : Q \rightarrow X$ defined on a dense subset $Q \supset \{0, d(x, y)\}$ of $[0, d(x, y)]$ and such that $f(0) = x$, $f(d(x, y)) = y$, and $\omega_f \leq \omega$.

To prove that X is uniformly locally chain equi-connected in X_0 , fix any $\varepsilon > 0$ and find $0 < \delta < \varepsilon_0$ such that $\omega(\delta) < \varepsilon$. Next, given $\eta > 0$ find $l \in \mathbb{N}$ such that $\omega(\delta/l) < \eta$. Fix any points $x, y \in X_0$ with $d(x, y) < \delta$. It follows that there is a uniformly continuous function $f : Q \rightarrow X$ defined on a dense subset $Q \supset \{0, d(x, y)\}$ of $[0, d(x, y)]$ such that $f(0) = x$, $f(d(x, y)) = y$, and $\omega_f \leq \omega$. Since $d(x, y) < \delta$, we can find points $0 = t_0 \leq t_1 \leq \dots \leq t_l = d(x, y)$ in Q such that $t_i - t_{i-1} < \delta/l$. Letting $x_i = f(t_i)$ for $i \leq l$ we will get an η -chain x_0, \dots, x_l of length $\leq l$ linking the points $x = x_0$ and $y = x_l$. Since $|t_i - t_j| < \delta$, we get $d(x_i, x_j) = d(f(t_i), f(t_j)) \leq \omega_f(|t_i - t_j|) \leq \omega(\delta) < \varepsilon$, which means that this chain has diameter $< \varepsilon$. \square

Lemma 1 can be partly reversed.

Lemma 2. *A metric space (X, d) is uniformly locally trace equi-connected at a subset $X_0 \subset X$, provided X is uniformly locally chain equi-connected at some r -neighborhood $B(X_0, r)$ of X_0 .*

Proof. Assume that X is uniformly locally chain equi-connected at $B(X_0, r_0)$ for some $r_0 > 0$. To prove the lemma it suffices to find $\varepsilon_0 > 0$ and a continuity modulus ω such that for any points $x, y \in X_0$ with $d(x, y) < \varepsilon_0$ there is a uniformly continuous function $f : Q \rightarrow X$ defined on some dense subset $Q \supset \{0, d(x, y)\}$ of $[0, d(x, y)]$ and such that $\omega_f \leq \omega$, $f(0) = x$ and $f(d(x, y)) = y$. Having such a function $f : Q \rightarrow X$ and given any countable dense subset $Q' \supset \{0, d(x, y)\}$ of $[0, d(x, y)]$ we can find an increasing Lipschitz homeomorphism h of $[0, d(x, y)]$ with Lipschitz constant 2 such that $h(Q') = Q$. Then the composition $f \circ h|_{Q'} : Q' \rightarrow X$ is a uniformly continuous map such that $f \circ h(0) = x$, $f \circ h(d(x, y)) = y$ and $\omega_{f \circ h}(t) \leq \omega(2t)$ for all $t \geq 0$.

Using the definition of (ulcec) of X at X_0 we can construct a decreasing sequence $(\delta_n)_{n \in \mathbb{N}}$ of positive reals such that for each $n \in \mathbb{N}$ and $\eta > 0$ there is a number $l = l(n, \eta) \in \mathbb{N}$ such that any two points $x, y \in U$ with $d(x, y) < \delta_n$ can be linked by an η -chain of diameter $< 2^{-n}r_0$ and length $\leq l$. Without loss of generality, $\delta_n \leq 2^{-n}r_0$ for all n .

Now for every $n \in \mathbb{N}$ take any $l_n \geq l(n, \delta_{n+1})$ and let $\Delta_n = 1/(l_1 \cdots l_n)$. Let also $\Delta_0 = 1$. Replacing (l_n) by a larger sequence, if necessary, we can assume that $\Delta_n < \delta_{n+2}$ for all $n \in \mathbb{N}$.

For every $m \in \mathbb{N}$ consider the finite subset

$$Q_m = \left\{ \sum_{k=1}^m \frac{i_k}{\Delta_k} : 0 \leq i_k < l_k \text{ for all } k \leq m \right\}$$

of $[0, 1)$ and let $Q = \bigcup_{m \geq 1} Q_m$. It is clear that Q is a countable dense subset of $[0, 1]$.

A continuity modulus ω will be defined as the supremum $\omega = \sup_{n \in \mathbb{N}} \omega_n$ of the sequence (ω_n) of continuity moduli defined recursively: $\omega_0 \equiv 0$ and

$$\omega_{n+1}(t) = \begin{cases} 0 & \text{if } t < \Delta_{n+1}; \\ 2^{-n}r_0 + \omega_n(t) & \text{if } t \geq \Delta_{n+1} \end{cases}$$

for $n \in \mathbb{N}$. It is easy to see that each function ω_n is non-decreasing and the supremum $\omega = \sup_{n \in \mathbb{N}} \omega_n$ is a well-defined continuity modulus.

To finish the proof it rests to verify that $\varepsilon_0 = \delta_2$ and ω satisfy our requirements. Take any two points $x, y \in X_0$ with $d(x, y) < \varepsilon_0 = \delta_2$. Find $k \geq 2$ with $\delta_{k+1} \leq d(x, y) < \delta_k$. Then $\Delta_{k-1} < \delta_{k+1} \leq d(x, y)$.

By induction we shall construct a sequence of functions $\{f_m : Q_m \rightarrow B(X_0, r_0)\}_{m \geq k-1}$ such that the following conditions are satisfied for every $m \geq k-1$:

- 1) $f_m(q) = y$ for all $q \in Q_m \cap [\Delta_{k-1}, 1]$;
- 2) $f_{m+1}|_{Q_m} = f_m$;
- 3) $f_m(Q_m) \subset B(X_0, r_0 \sum_{i=k}^m 2^{-i})$;
- 4) $d(f_m(p), f_m(q)) < \delta_{m+1}$ for any neighbor points $p, q \in Q_m$;
- 5) $\text{diam} f_m(Q_m \cap [p, q]) < 2^{-m} r_0$ for any neighbor points p, q of Q_{m-1} .
- 6) $\omega_{f_m} \leq \omega_m$.

Two points p, q of a finite subset $F \subset [0, 1]$ are called *neighbor points* if (p, q) is a connected component of $[0, 1] \setminus F$.

To start the inductive construction let $f_{k-1}(0) = x$ and $f_{k-1}(q) = y$ for any $q \in Q_{k-1} \cap [\Delta_{k-1}, 1]$. Assume that for some $m \geq k - 1$ functions f_{k-1}, \dots, f_m satisfying the conditions (1)–(6) have been constructed. Take any neighbor points p, q of Q_m . If $\min\{p, q\} \geq \Delta_{k-1}$ let $f_{m+1}|_{[p, q]} \cap Q_{m+1} = \{y\}$. Otherwise use the conditions (3) and (4) to conclude that the points $f_m(p), f_m(q)$ belong to $B(X_0, r_0)$ and satisfy $d(f_m(p), f_m(q)) < \delta_{m+1}$. The choice of the sequences (δ_i) and (l_i) guarantees the existence of a δ_{m+2} -chain $f_m(p) = c_0, \dots, c_{l_{m+1}} = f_m(q)$ of diameter $< r_0/2^{m+1}$ and length l_{m+1} connecting the points $f_m(p)$ and $f_m(q)$. The intersection $(p, q) \cap Q_{m+1}$ contains exactly $m_{m+1} - 1$ points, so we can assign to each of these points a point c_i from the chain to satisfy the condition (4). Since $\text{diam} f_{m+1}(Q_{m+1} \cap [p, q]) < 2^{-(m+1)} r_0$, we see that the conditions (3,5) are satisfied.

To show that $\omega_{f_{m+1}} \leq \omega_{m+1}$ it suffices to verify that, $d(f_{m+1}(p), f_{m+1}(q)) \leq \omega_{m+1}(|p - q|)$ for any two distinct points $p, q \in Q_{m+1}$. Given such points $p, q \in Q_{m+1}$ find points $p', q' \in Q_m$ with $\max\{|p - p'|, |q - q'|\} < \Delta_m$ and $|p' - q'| \leq |p - q|$. Then

$$\begin{aligned} d(f_{m+1}(p), f_{m+1}(q)) &\leq d(f_{m+1}(p), f_m(p')) + d(f_{m+1}(q), f_m(q')) + d(f_m(p'), f_m(q')) \leq \\ &\leq 2 \cdot 2^{-(m+1)} r_0 + \omega_m(|p' - q'|) \leq 2^{-m} r_0 + \omega_m(|p - q|) = \omega_{m+1}(|p - q|). \end{aligned}$$

This completes the inductive step. Now let $f = \bigcup_{m \in \mathbb{N}} f_m : Q \rightarrow B(X_0, r_0)$ and observe that $f(0) = x$, $f(\Delta_{k-1}) = \lim_{q \rightarrow d(x, y)} f(q) = y$ and $\omega_f \leq \sup \omega_m = \omega$. \square

Lemma 3. *A metric space (X, d) is uniformly locally chain equi-connected, provided X is chain connected im kleinen and the uniformity of X is generated by a metric d which is ω -convex on each chain connected component of X for some continuity modulus ω .*

Proof. To show that X is uniformly locally chain equi-connected, fix arbitrary $\varepsilon > 0$. Replacing ε by a smaller positive number we can assume that any two points $x, y \in X$ with $d(x, y) < \varepsilon$ belong to the same chain connected component of X .

For this $\varepsilon > 0$ find $a > 0$ such that $\omega(a) < \frac{\varepsilon}{2}$ and let $\delta = \omega(a)$. Given $\eta > 0$ find $l \in \mathbb{N}$ such that $\omega(a/l) < \eta$.

Now take any points $x, y \in X$ with $d(x, y) < \delta = \omega(a) < \frac{\varepsilon}{2}$. They belong to some chain connected component C on which the metric d is ω -convex. Let $x_0 = x$. Since d is ω -convex on C and $d(x, y) < \delta = \omega(a) = \omega(\frac{a}{l} + \frac{a(l-1)}{l})$ there is a point $x_1 \in C$ such that $d(x_0, x_1) < \omega(a/l) < \eta$ and $d(x_1, y) < \omega(\frac{a(l-1)}{l})$. Applying the ω -convexity of d on C once more, we shall find a point $x_2 \in C$ such that $d(x_1, x_2) < \omega(a/l) < \eta$ and $d(x_2, y) < \omega(\frac{a(l-2)}{l})$. Proceeding in this way we shall construct a sequence $x = x_0, x_1, \dots, x_l = y$ of points of C such that $d(x_i, x_{i+1}) < \omega(a/l) < \eta$ and $d(x_i, y) < \frac{a(l-i)}{l}$ for all $i < l$. It is clear that x_0, \dots, x_l is an η -chain of length l and diameter $< 2\omega(a) < \varepsilon$ linking the points x and y . \square

In the proof of convexification Lemma 5 we will exploit the following elementary fact which can be proven by a simple geometric argument.

Lemma 4. *If f is a concave continuity modulus, then*

- (1) $f(b + d) - f(b) \leq f(a + d) - f(a)$ for any $0 \leq a \leq b$ and $d \geq 0$;
- (2) $f(s + t) - f(s) - f(t) \leq f(a + b) - f(a) - f(b)$ for any $0 \leq a \leq s$ and $0 \leq b \leq t$.

Lemma 5. *If a metric space (X, d) is uniformly locally chain equi-connected, then X is weakly convexifiable.*

Proof. If X is (ulcec), then it is chain connected im kleinen and is (ultec) by Lemma 2. Consequently there are $R > 0$ and a continuity modulus ω such that any points $x, y \in X$ with $d(x, y) \leq R$ can be linked by a uniformly continuous function $f : Q \rightarrow X$ defined on a dense subset $Q \supset \{0, d(x, y)\}$ of $[0, d(x, y)]$ and such that $f(0) = x$, $f(d(x, y)) = y$ and $\omega_f \leq \omega$.

Let C be the closed convex hull of the set $\{(t, \omega(t)) : t \in (0, R]\} \cup \{(nR, (n+2)\omega(R)) : n \in \mathbb{N}\}$ in the real plane \mathbb{R}^2 . It can be easily shown that the function $\gamma(t) = \max\{y \in C : (t, y) \in C\}$ is a concave continuity modulus with $\gamma(t) \geq \omega(t)$ for all $t \leq R$ and $\gamma(iR) \geq (i+2)\omega(R)$ for all $i \in \mathbb{N}$. Then 2γ is a concave continuity modulus as well.

Given a concave continuity modulus $\alpha \geq 2\gamma$ we shall define a metric ρ on X as follows. For a pair of points $(x, y) \in X$ let $R(x, y)$ be the set of all positive real numbers r for which there is a uniformly continuous map $f : Q \rightarrow X$ from a countable dense subset $Q \supset \{0, r\}$ of $[0, r]$ such that $f(0) = x$, $f(r) = y$ and $\omega_f \leq \alpha$. It is easy to see that $R(x, y) = R(y, x)$ for all $x, y \in X$.

Let us show that $R(x, y) \neq \emptyset$ for any points x, y from the same chain connected component. Given such points x, y find an R -chain $x = x_0, \dots, x_l = y$ linking points x, y . For every $i \leq l$ we get $d(x_{i-1}, x_i) < R$. Consequently, we can find a uniformly continuous function $f_i : Q_i \rightarrow X$ defined on a countable dense subset $Q_i \supset \{(i-1)R, iR\}$ of the interval $[(i-1)R, iR]$ such that $f_i((i-1)R) = x_{i-1}$, $f_i(iR) = x_i$ and $\omega_{f_i} \leq \omega$. Let $Q = \bigcup_{i \leq l} Q_i$ and $f = \bigcup_{i \leq l} f_i : Q \rightarrow X$. It is clear that f is a uniformly continuous function with $f(0) = x$ and $f(lR) = y$.

Let us show that $\omega_f(t) \leq \alpha(t)$ for all $t \geq 0$. Take any points $q < q'$ in Q with $|q - q'| \leq t$. Fix numbers $i \leq j \leq l$ such that $(i-1)R < q \leq iR$ and $(j-1)R \leq q' < jR$. If $i = j$, then $d(f(q), f(q')) \leq \omega_{f_i}(t) \leq \omega(t) \leq \gamma(t) \leq \alpha(t)$. If $j = i+1$, then

$$\begin{aligned} d(f(q), f(q')) &\leq d(f_i(q), f_i(iR)) + d(f_j(iR), f_j(q')) \leq \\ &\leq \omega_{f_i}(iR - q) + \omega_{f_j}(q' - iR) \leq 2\omega(t) \leq 2\gamma(t) \leq \alpha(t). \end{aligned}$$

If $j > i+1$, then

$$(j-i-1)R \leq q' - q = (q' - (j-1)R) + (j-i-1)R + (iR - q) < 2R + (j-i-1)R = (j-i+1)R$$

and

$$\begin{aligned} d(f(q), f(q')) &\leq d(f(q), f(iR)) + d(f(iR), f((i+1)R)) + \dots + d(f((j-1)R), f(q')) \leq \\ &\leq (j-i+1)\omega(R) \leq \gamma((j-i-1)R) \leq \gamma(q' - q) \leq \gamma(t) \leq \alpha(t). \end{aligned}$$

For points $x, y \in X$ let $r(x, y) = \inf(\{\infty\} \cup R(x, y))$. Observe that $r(x, y) = r(y, x)$ and $r(x, y) < \infty$ if and only if the points x, y belong to the same chain connected component of X . It follows from the definition of r that for any $x, y \in X$ we get $d(x, y) \leq \alpha \circ r(x, y)$. Moreover, $r(x, y) \leq d(x, y)$ if $d(x, y) \leq R$.

Define a function $p : X \rightarrow [0, \infty]$ letting

$$p(x, y) = \inf \left\{ \sum_{i=1}^m \alpha \circ r(x_{i-1}, x_i) : x = x_0, \dots, x_m = y \right\}.$$

It is clear that this function is symmetric and satisfies the triangle inequality (we assume that $x + \infty = \infty \geq x$ for any $x \in [0, \infty]$). Also $p(x, y) \leq \alpha \circ r(x, y) \leq \alpha \circ d(x, y)$ if $d(x, y) \leq R$.

On the other hand, for any sequence $x = x_0, \dots, x_m = y$ in X we get

$$\sum_{i=1}^m \alpha \circ r(x_{i-1}, x_i) \geq \sum_{i=1}^m d(x_{i-1}, x_i) \geq d(x_0, x_m) = d(x, y)$$

which implies that $d(x, y) \leq p(x, y)$, and $p(x, y) \leq \alpha \circ d(x, y)$ if $d(x, y) \leq R$.

Take any subset $S \subset X$ meeting each chain connected component of X in a unique point. Define a metric ρ on X letting $\rho(x, y) = p(x, y)$ if x, y belong to the same connected component and $\rho(x, y) = 2R + p(x, x_0) + p(y, y_0)$ where $x_0 \in S$ (resp. $y_0 \in S$) belongs to the chain connected component of x (resp. y). It is easy to see that ρ is a metric. Moreover, $d(x, y) \leq \rho(x, y) = p(x, y) \leq \alpha \circ d(x, y)$ if $d(x, y) < R$. This means that the metric ρ is uniformly equivalent to d and thus generates the uniformity of X .

It rests to verify that ρ is α -convex on each connected component C of X . Fix any points $x, y \in C$ and pick positive real numbers s, t with $\rho(x, y) < \alpha(s+t)$. If $\rho(x, y) < \alpha(s)$, then put

$z = y$ and note that $\rho(x, z) = \rho(x, y) < \omega(s)$ while $\rho(z, y) = 0 < \omega(t)$. So further we assume that $\rho(x, y) \geq w(s)$.

By the definition of $p(x, y) = \rho(x, y)$ there is a chain $x = x_0, x_1, \dots, x_m = y$ such that $\rho(x, y) \leq \sum_{i=1}^m \alpha \circ r(x_{i-1}, x_i) < \alpha(s + t)$. Let $j \geq 0$ be the largest number such that $\sum_{i=1}^j \alpha \circ r(x_{i-1}, x_i) < \alpha(s)$. The inequality $\rho(x, y) \geq \alpha(s)$ implies that $j < l$.

Let $A = \sum_{i=1}^j \alpha \circ r(x_{i-1}, x_i)$, $B = \sum_{i=j+2}^l \alpha \circ r(x_{i-1}, x_i)$, and fix any $\Delta > r(x_{j-1}, x_j)$ with

$$(1) \quad A + \alpha(\Delta) + B < \alpha(s + t).$$

It follows from the definition of $r(x_{j-1}, x_j) < \Delta$ that there is a uniformly continuous function $f : Q \rightarrow X$ defined on a countable dense subset $Q \supset \{0, \Delta\}$ of $[0, \Delta]$ and such that $f(0) = x_{j-1}$, $f(\Delta) = x_j$ and $w_f \leq \alpha$.

By the choice of j , we get $A < \alpha(s)$, and $A + \alpha(\Delta) \geq \alpha(s)$. Using the continuity of α find $\delta \in [0, \Delta]$ such that $A + \alpha(\delta) = \alpha(s)$. It follows that $\delta \leq s$ and

$$(2) \quad B < \alpha(s + t) - \alpha(\Delta) - A = \alpha(s + t) - \alpha(\Delta) - (\alpha(s) - \alpha(\delta)).$$

We claim that $\Delta - \delta < t$. Assuming the converse we would get

$$(3) \quad \alpha(\Delta) - \alpha(\delta) = \alpha(\delta + (\Delta - \delta)) - \alpha(\delta) \geq \alpha(\delta + t) - \alpha(\delta) \geq \alpha(s + t) - \alpha(s).$$

The last inequality follows from $\delta \leq s$ and Lemma 4(1). Combining (1-3) we will get $\alpha(s + t) > A + \alpha(\Delta) \geq (\alpha(s) - \alpha(\delta)) + \alpha(\Delta) \geq (\alpha(s) - \alpha(\delta)) + \alpha(s + t) - \alpha(s) + \alpha(\delta) = \alpha(s + t)$, which is a contradiction. Thus $\delta \leq s$ and $\Delta - \delta < t$ and we can apply Lemma 4(2) to conclude that $\alpha(\delta) + \alpha(\Delta - \delta) - \alpha(\Delta) \leq \alpha(s) + \alpha(t) - \alpha(s + t)$ and hence

$$\alpha(\Delta - \delta) \leq (\alpha(s) + \alpha(t) - \alpha(s + t)) + (\alpha(\Delta) - \alpha(\delta)).$$

Adding to this inequality the inequality (2), we will get

$$\begin{aligned} \alpha(\Delta - \delta) + B &< \alpha(\Delta - \delta) + (\alpha(s + t) - \alpha(s)) - (\alpha(\Delta) - \alpha(\delta)) \leq \\ &\leq (\alpha(s) + \alpha(t) - \alpha(s + t)) + (\alpha(\Delta) - \alpha(\delta)) + \\ &+ (\alpha(s + t) - \alpha(s)) - (\alpha(\Delta) - \alpha(\delta)) = \alpha(t). \end{aligned}$$

By the continuity of α find $q \in Q \cap [0, \delta]$ such that $\alpha(\Delta - q) + B < \alpha(t)$. Then $A + \alpha(q) < A + \alpha(\delta) = \alpha(s)$ and $B + \alpha(\Delta - q) < \alpha(t)$. It can be shown that for the point $z = f(q)$ we get $\rho(x, z) = p(x, z) \leq A + \alpha(q) < \alpha(s)$ and $\rho(z, y) = p(y, z) \leq B + \alpha(\Delta - q) < \alpha(t)$. This completes the proof of the α -convexity of the metric ρ on the chain connected component C . \square

2. LAWSON SEMILATTICES AND (UNIFORM) ANR'S

By a *topological semilattice* we understand a pair (X, \vee) consisting of a topological space X and a continuous associative commutative idempotent operation $\vee : X \times X \rightarrow X$. A topological semilattice (X, \vee) is called a *Lawson semilattice* if X has a base of the topology consisting of subsemilattices. The following observation made in [BKS] allows us to reduce the study of the ANR-property in Lawson semilattices to verifying the local path connectedness.

Proposition 1. *A metrizable Lawson semilattice L is an ANR (an AR) if and only if L is locally path-connected (and connected).*

This result has a uniform counterpart. By a *Lawson metric semilattice* we shall understand a metric space (X, d) endowed with a semilattice operation $\vee : X \times X \rightarrow X$ such that $d(x \vee y, x' \vee y') \leq \max\{d(x, x'), d(y, y')\}$ for all $x, y, x', y' \in X$. It is easy to see that each Lawson metric semilattice is Lawson as a topological semilattice. The following important result proven in [KSY] is a uniform counterpart of Proposition 1.

Proposition 2. *A Lawson metric semilattice X is a uANR (and uAR) if and only if it is uniformly locally path-connected (and chain connected).*

It is easy to see that for any bounded metric space (X, d) the hyperspace $(2^X, d_H, \cup)$ is a Lawson metric semilattice with respect to the operation of union \cup . Hence 2^X is a uANR (and a uAR) if and only if it is uniformly locally path-connected (and chain connected).

Propositions 1, 2 show that for Lawson metric semilattices the implications (ANR) \Rightarrow (lpc) and (uANR) \Rightarrow (ulpc) can be reversed. We do not know if this can be done for the implication (ANUR) \Rightarrow (ulpec).

Problem 4. *Suppose that a Lawson metric semilattice X is uniformly locally path equi-connected. Is X an absolute neighborhood uniform retract? Is 2^X an ANUR?*

3. ANR-PROPERTY OF HYPERSPACES $\mathcal{D}(X)$

The aim of this section is to prove the implication (10) \Rightarrow (11) of Theorem 1. We shall say that a metric space (X, d) has *totally bounded small balls* if there is $r > 0$ such that each ball $B(x, r)$, $x \in X$, is totally bounded. It is clear that each uniformly locally compact space has totally bounded small balls.

Proposition 3. *Let (X, d) be a uniformly locally path equi-connected metric space. If X has totally bounded small balls, then the hyperspace $\mathcal{D}(X)$ is a uniform ANR.*

In light of Proposition 2 it suffices to prove that $\mathcal{D}(X)$ is uniformly locally path connected. The latter property of $\mathcal{D}(X)$ can be easily derived from the following two lemmas, first of which belongs to mathematical folklore.

Lemma 6. *A metric space (M, d) is uniformly locally path connected provided there is a dense subset $D \subset M$ such that for any $\varepsilon > 0$ there is $\delta > 0$ such that any two points $x, y \in D$ with $d(x, y) < \delta$ can be linked by a path $f : [0, 1] \rightarrow M$ with diameter $< \varepsilon$.*

To apply this Lemma with $M = \mathcal{D}(X)$ and $D = u\mathcal{D}(X)$ we need

Lemma 7. *Let (X, d) be a uniformly locally path equi-connected metric space having totally bounded small balls. For any $\varepsilon > 0$ there is $\delta > 0$ such that any two sets $A, B \in u\mathcal{D}(X)$ with $d_H(A, B) < \delta$ can be linked by a path $f : [0, 1] \rightarrow \mathcal{D}(X)$ with diameter $< \varepsilon$.*

Proof. Since X is uniformly locally path equi-connected, there are $r > 0$ and a continuity modulus ω such that any two points $x, y \in X$ with $d(x, y) < r$ can be connected by a path $f : [0, d(x, y)] \rightarrow X$ such that $f(0) = x$, $f(d(x, y)) = y$ and $\omega_f \leq \omega$.

We can assume that r is so small that the ball $B(x, 2\omega(r))$ is totally bounded for each $x \in X$.

Given arbitrary $\varepsilon > 0$ find positive $\delta < r$ such that $\omega(\delta) < \varepsilon$. Fix any uniformly discrete sets $A, B \in u\mathcal{D}(X)$ with $\rho = d_H(A, B) < \delta$.

For each points $a \in A$ select a point $b_a \in B$ with $d(a, b_a) \leq \rho$ and for any $b \in B$ find $a_b \in A$ with $d(b, a_b) \leq \rho$. By the choice of r and ω , there are maps $f_a, f_b : [0, \rho] \rightarrow X$ such that $f_a(0) = a$, $f_a(\rho) = b_a$, $f_b(0) = a_b$, $f_b(\rho) = b$, and $\omega_{f_a} \leq \omega$, $\omega_{f_b} \leq \omega$. Define a map $f : [0, \rho] \rightarrow \mathcal{D}(X)$ letting $f(t) = \{f_a(t), f_b(t) : a \in A, b \in B\}$ for $t \in [0, \rho]$. Let us show that f is well-defined, i.e, for each $t \in [0, \rho]$ the set $f(t)$ is closed and discrete in X .

Given any point $x_0 \in X$ it suffices to find a neighborhood of x_0 having finite intersection with $f(t)$. For this consider the sets $A' = \{a \in A : d(f_a(t), x_0) < \omega(r)\}$ and $B' = \{b \in B : d(f_b(t), x_0) < \omega(r)\}$. Since $d(f_a(t), a) \leq \omega(\rho) \leq \omega(r)$ for all $a \in A$, the set A' lies in the ball $B(x_0, 2\omega(r))$. Now the uniform discreteness of $A \supset A'$ and the total boundedness of $B(x_0, 2\omega(r))$ imply that the set A' is finite. By analogy, we can prove that the set B' is finite. Consequently, $B(x_0, \omega(r)) \cap f(t)$ has size $\leq |A'| + |B'|$, which just yields that $f(t)$ is closed and discrete.

It is easy to see that the path $f : [0, \rho] \rightarrow \mathcal{D}(X)$ links the sets A and B and satisfies $\omega_f \leq \omega$. This yields also that $\text{diam} f([0, \rho]) \leq \omega(\rho) \leq \omega(\delta) < \varepsilon$. \square

4. LOCAL CHAIN CONNECTEDNESS IN HYPERSPACES

In this section we will prove the implications (6) \Rightarrow (7) of Theorem 1 and (4) \Rightarrow (5) of Theorem 3. Our proof will exploit the famous Ramsey Theorem asserting that for each finite coloring $\chi : [\mathbb{N}]^2 \rightarrow \{0, \dots, r\}$ of the collection $[\mathbb{N}]^2$ of two-element subsets of \mathbb{N} there is an infinite subset $I \subset \mathbb{N}$ such that the set $[I]^2$ is monochromatic, that is $\chi([I]^2) \equiv \text{const}$, see [GRS, §1.5].

A sequence $(x_n)_{n=1}^\infty$ of points of a set X will be called *regular* if either $|\{x_n : n \in \mathbb{N}\}| = 1$ or else $x_n \neq x_m$ for distinct $n, m \in \mathbb{N}$.

Lemma 8. *A metric space (X, d) fails to be uniformly locally chain equi-connected at a subset $B \subset X$ if and only if $\exists \varepsilon > 0 \forall \delta > 0 \exists \eta > 0 \exists \{x_i, y_i : i \in \mathbb{N}\} \subset B$ such that*

- (1) *for all $i \in \mathbb{N}$ $d(x_i, y_i) < \delta$ and x_i, y_i cannot be connected by an η -chain in X of length $\leq i$ and diameter $< \varepsilon$;*
- (2) *the sets $\{x_i : i \in \mathbb{N}\}$, $\{y_i : i \in \mathbb{N}\}$ are η -separated;*
- (3) *the sequences (x_i) and (y_i) are regular.*

Proof. The “if” part is trivial. To prove the “only if” part, assume that X fails to be uniformly locally chain equi-connected at B . This means that for some $\varepsilon_0 > 0$ and each $\delta > 0$ there is $\eta > 0$ such that for each $n \in \mathbb{N}$ there is a pair of points $a_n, b_n \in B$ with $d(a_n, b_n) < \delta$ that cannot be linked by an η -chain of length $\leq n$ and diameter $\leq \varepsilon_0$. Without loss of generality we can assume that $\eta < \varepsilon_0/4$. Let $\varepsilon = \varepsilon_0/2$.

Define a coloring of $[\mathbb{N}]^2$ into four colors as follows. Colorate a two-element subset $\{i, j\} \subset \mathbb{N}$ in

- *black* if $d(a_i, a_j) < \eta$ and $d(b_i, b_j) < \eta$;
- *white* if $d(a_i, a_j) \geq \eta$ and $d(b_i, b_j) \geq \eta$;
- *green* if $d(a_i, a_j) < \eta$ and $d(b_i, b_j) \geq \eta$;
- *blue* if $d(a_i, a_j) \geq \eta$ and $d(b_i, b_j) < \eta$.

Applying the Ramsey Theorem, we can find an infinite subset $N \subset [2, \infty) \cap \mathbb{N}$ such that all two-element subsets of N are colored by the same color $\chi \in \{\text{black, white, green, blue}\}$. Let $N = \{n_k : k \in \mathbb{N}\}$ be an increasing enumeration of N and note that $n_k > k$ for all k . Fix any number $m \in N$.

1. Assuming that the color χ is black, let $x_i = a_m, y_i = b_m$ for all $i \in N$. Then the conditions (2), (3) of Lemma 8 is trivially satisfied.

To verify the condition (1), assume that for some $i \in \mathbb{N}$ the points $x_i = a_m$ and $y_i = b_m$ can be connected by a η -chain of diameter $< \varepsilon = \varepsilon_0/2$ and length $\leq i$. Take any $n \in N$ with $n > i + 2$. Since $\max\{d(a_m, a_n), d(b_m, b_n)\} < \eta < \varepsilon_0/4$, the points a_n and b_n can be connected by an η -chain of length $\leq i + 2 \leq n$ and diameter $< \frac{\varepsilon_0}{2} + 2\frac{\varepsilon_0}{4} = \varepsilon_0$, which contradicts to the choice of these points.

2. Assuming that the color χ is white let $x_i = a_{n_i}$ and $y_i = b_{n_i}$ for all $i \in \mathbb{N}$ and observe that the sequences $(x_i), (y_i)$ satisfy the requirements of the lemma.

3. Assuming that the color χ is green let $x_i = a_m$ and $y_i = b_{n_i}$ for all $i \in \mathbb{N}$. It is clear that the sequences $(x_i), (y_i)$ satisfy the condition (2) and (3). Assume that for some $i \in \mathbb{N}$ the points $x_i = a_m$ and $y_i = b_{n_i}$ can be connected by an η -chain of length $\leq i$ and diameter $\leq \varepsilon_0/2$. Since $d(a_m, a_{n_i}) < \eta < \varepsilon_0/2$, the points a_{n_i}, b_{n_i} can be connected by an η -chain of length $\leq i + 1 \leq n_i$ and diameter $\leq \frac{\varepsilon_0}{2} + \eta \leq \varepsilon_0$, which contradicts to the choice of these points.

4. Assuming that the color χ is blue let $x_i = a_{n_i}$ and $y_i = b_m$ for all $i \in \mathbb{N}$. By analogy with the previous case we can show that the sequences (x_i) and (y_i) satisfy the requirements of the lemma. \square

Lemma 9. *A metric space (X, d) is uniformly locally chain equi-connected at a closed subset $B \subset X$, provided the hyperspace 2^X is locally chain connected at $\mathcal{D}_0(B)$.*

Proof. Assume that some closed subset B of X fails to be uniformly locally chain equi-connected. Applying Lemma 8, for some $\varepsilon_0 > 0$ and each $n \in \mathbb{N}$ we can find $\eta_n > 0$ and regular sequences $(x_{n,k})_{k \in \mathbb{N}}, (y_{n,k})_{k \in \mathbb{N}}$ of points in B such that $d(x_{n,k}, y_{n,k}) < \frac{1}{n}$, the sets $\{x_{n,k} : k \in \mathbb{N}\}$ and $\{y_{n,k} : k \in \mathbb{N}\}$ are η_n -separated and for each $k \in \mathbb{N}$ the points $x_{n,k}, y_{n,k}$ cannot be connected by an η_n -chain of length $\leq k$ and diameter $\leq \varepsilon_0$. Without loss of generality, we may assume that $\eta_n < \min\{\eta_{n-1}, \frac{1}{n}\}$.

For every $n \in \mathbb{N}$ find an infinite subset $I_n \subset \mathbb{N}$ such that

$$\text{diam}\{x_{n,k}, y_{n,k} : k \in I_n\} < \eta_n + \inf\{\text{diam}\{x_{n,k}, y_{n,k} : k \in I\} : I \subset \mathbb{N}, |I| = \infty\}.$$

Let $A_n = \{x_{n,k}, y_{n,k} : k \in I_n\}$, $n \in \mathbb{N}$. Now consider separately two cases.

I. $\inf_{n \in \mathbb{N}} \text{diam} A_n > 0$. In this case

$$\Delta = \inf\{\text{diam}\{x_{n,k}, y_{n,k} : k \in I\} : I \subset \mathbb{N}, |I| = \infty\} > 0$$

and for any subset $D \subset X$ of diameter $< \Delta$ the set $\{k \in \mathbb{N} : \{x_{n,k}, y_{n,k}\} \subset D\}$ is finite for each n . The condition $\lim_{n \rightarrow \infty} \eta_n = 0 < \inf_{n \in \mathbb{N}} \text{diam} A_n$ and the regularity of the sequences $(x_{n,k})_{k \in \mathbb{N}}$, $(y_{n,k})_{k \in \mathbb{N}}$ imply that the sets A_n are infinite for all n exceeding some number n_0 .

It follows that for any finite subset $F \subset X$ and any $n > n_0$ the complement $A_n \setminus B(F, \Delta/3)$ is infinite. Using this observation we shall construct a $\Delta/3$ -separated subset $D_0 \subset \bigcup_{n \in \mathbb{N}} A_n \subset B$ such that the intersection $D_0 \cap A_n$ is infinite for each $n > n_0$. The construction of D_0 looks as follows. Let $\xi : \mathbb{N} \rightarrow (n_0, \infty) \cap \mathbb{N}$ be a function such that $\xi^{-1}(n)$ is infinite for all $n > n_0$. By induction for every $n \in \mathbb{N}$ select a point $d_n \in A_{\xi(n)} \setminus B(\{d_1, \dots, d_{n-1}, \Delta/3\})$. Then the set $D_0 = \{d_n : n \in \mathbb{N}\}$ has the desired property.

Since 2^X is locally chain connected at $\mathcal{D}_0(B) \ni D_0$, for the number $\varepsilon = \min\{\frac{\Delta}{9}, \frac{\varepsilon_0}{2}\}$ there is $\delta > 0$ such that each countable uniformly discrete subset $D \subset B$ with $d_H(D_0, D) < \delta$ for any $\eta > 0$ can be connected with D_0 by an η -chain of diameter $< \varepsilon$. Take any number $n > n_0$ with $\frac{1}{n} < \min\{\frac{\Delta}{9}, \delta\}$ and consider two countable uniformly discrete sets:

$$\begin{aligned} D_x &= D_0 \cup \{x_{n,k} \in A_n : \{x_{n,k}, y_{n,k}\} \cap D_0 \neq \emptyset\} \\ D_y &= D_0 \cup \{y_{n,k} \in A_n : \{x_{n,k}, y_{n,k}\} \cap D_0 \neq \emptyset\} \end{aligned}$$

It follows from $d(x_{n,k}, y_{n,k}) < \frac{1}{n} < \delta$ that $d_H(D_0, D_x) \leq \frac{1}{n} < \delta$ and consequently, D_0 can be linked with D_x by an η_n -chain $D_0, D_1, \dots, D_p = D_x$ of diameter $< \varepsilon$. By analogy, D_0 can be connected with D_y by an η_n -chain $D_0 = E_0, E_1, \dots, E_q = D_y$ of diameter $< \varepsilon$.

Find a number $l > p + q$ such that $\{x_{n,l}, y_{n,l}\} \cap D_0$ contains some point z (such a number l exists because the intersection $D_0 \cap A_n$ is infinite). Since $x_{n,l} \in D_x$, selecting suitable points in the η_n -chain D_0, D_1, \dots, D_p we can construct an η_n -chain $x_{n,l} = z_0, \dots, z_p$ with $z_i \in D_{p-i}$ for $i \leq p$. We claim that $z_p = z$ and $\{z_0, \dots, z_p\} \subset B(z, \varepsilon_0/2)$. Indeed, $\{z_0, \dots, z_p\} \subset D_0 \cup \dots \cup D_p \subset B(D_0, \varepsilon) \subset B(D_0, \varepsilon_0/2)$. Taking into account that $\varepsilon \leq \frac{\Delta}{9}$, the set D_0 is $\Delta/3$ -separated, $d(z_0, z) < \frac{1}{n} < \frac{\Delta}{9}$, and $d(z_{i-1}, z_i) < \eta_n < \frac{1}{n} < \frac{\Delta}{9}$ for $i \leq p$, we conclude that $\{z_0, \dots, z_p\} \subset B(z, \varepsilon)$ and thus $z_p \in D_0 \cap B(z, \varepsilon) = \{z\}$ which just yields $z_p = z$.

By analogy, using the η_n -chain E_0, \dots, E_q between D_0 and D_y , we can construct an η_n -chain $z_p, z_{p+1}, \dots, z_{p+q} \in B(z, \varepsilon)$ connecting z with $y_{n,l}$. Then $z_0, \dots, z_{p+q} \in B(z, \varepsilon)$ is an η_n -chain of length $p + q < l$ and diameter $< 2\varepsilon \leq \varepsilon_0$ linking the points $x_{n,l}$ and $y_{n,l}$, which contradicts to their choice. This completes the proof in the case $\Delta > 0$.

II. Now we assume that $\inf_{n \in \mathbb{N}} \text{diam} A_n = 0$. Take an infinite subset $N \subset \mathbb{N}$ such that $\lim_{n \in N} \text{diam} A_n = 0$. In each set A_n , $n \in N$, fix a point $a_n \in A_n$. We consider separately three subcases.

II.1. The set $\{a_n : n \in N\}$ is not totally bounded in X . Then, replacing N by a smaller subset, if necessary, we can assume that the set $D_0 = \{a_n : n \in N\}$ is α -separated for some $\alpha > 0$. Since 2^X is locally chain connected at $\mathcal{D}_0(B) \ni D_0$, for the number $\varepsilon = \min\{\frac{\alpha}{3}, \frac{\varepsilon_0}{2}\}$ there is $\delta > 0$ such that each countable uniformly discrete subset $D \subset B$ with $d_H(D_0, D) < \delta$ can be connected with D_0 in 2^X by an η -chain of diameter $< \varepsilon$ for every $\eta > 0$. Take any $n \in N$ with $\text{diam} A_n < \min\{\frac{\alpha}{3}, \delta\}$ and consider two countable uniformly discrete sets:

$$D_x = D_0 \cup \{x_{n,i} : i \in I_n\} \text{ and } D_y = D_0 \cup \{y_{n,i} : i \in I_n\}.$$

It follows from $\text{diam} A_n < \delta$ that $d_H(D_0, D_x) < \delta$ and consequently, D_0 can be linked with D_x and D_y by η_n -chains of diameter $< \varepsilon$ and length $\leq l$ for some l . Using these chains and repeating the argument from the case (I), for each $i \in I_n$ we can construct an η_n -chain of diameter $< \varepsilon_0$ and length $\leq 2l$ connecting the points $x_{n,i}, y_{n,i}$, which is not possible for $i > 2l$.

II.2. Next we consider the case when the sequence $\{a_n : n \in N\}$ has a cluster point $a_\infty \in B$. Replacing N by a smaller subset we can assume that the sequence $\{a_n : n \in N\}$ converges to a_∞ . Consider the one-point subset $D_0 = \{a_\infty\}$ and use the local chain connectedness of 2^X at $\mathcal{D}_0(B)$ in 2^X to find $\delta > 0$ such that any countable uniformly discrete subset $D \subset B$ with $d_H(D_0, D) < \delta$ can be linked with D_0 by an η -chain of diameter $\varepsilon_0/2$ for each $\eta > 0$. Find $n \in N$ such that $d(a_n, a_\infty) < \delta/2$ and $\text{diam} A_n < \delta/2$. Consider two countable bounded uniformly discrete sets:

$$D_x = \{x_{n,i} : i \in I_n\} \text{ and } D_y = \{y_{n,i} : i \in I_n\}.$$

It follows that $d_H(D_0, D_x) \leq d(a_n, a_\infty) + \text{diam}A_n < \delta$ and consequently, $D_0 = \{a_\infty\}$ can be linked with D_x and D_y by η_n -chains in 2^X of diameter $< \varepsilon_0/2$ and length $\leq l$ for some l . Using these chains for each $i \in I_n$ we can construct an η_n -chain in $B(a_\infty, \varepsilon_0/2)$ of length $\leq 2l$ connecting the points $x_{n,i}, y_{n,i}$, which is not possible for $i > 2l$.

II.3. Finally consider the case of a totally bounded sequence $\{a_n : n \in N\}$ having no limit points in X . Replacing N by a smaller subset, if necessary, we may assume that the sequence $\{a_n : n \in N\}$ is Cauchy and has diameter $< \varepsilon_0/8$. With help of the Ramsey theorem we select from this sequence an especially nice subsequence as follows. Colorate a two-point subset $\{n, m\} \in [N]^2$ with $n < m$ in

- *white* if a_n and a_m can be connected by an η_m -chain of diameter $< \varepsilon_0/4$;
- *black* otherwise.

Apply the Ramsey Theorem to find an infinite subset $M \subset N$ such that all two-element subsets of M have the same color. We claim that this color is white. For this it suffices to find a two-element subset $\{n, m\} \subset M$ colored in white.

Consider the totally bounded closed discrete subset $D_0 = \{a_n : n \in M\} \in \mathcal{D}_0(B)$. Since 2^X is locally chain connected at $\mathcal{D}_0(B)$, for the real number $\frac{\varepsilon_0}{16}$ there is $\delta > 0$ such that each finite subset $D \subset B$ with $d_H(D_0, D) < \delta$ can be connected with D_0 by an η -chain of diameter $< \frac{\varepsilon_0}{16}$ for each $\eta > 0$. Since $D_0 = \{a_n : n \in M\}$ is totally bounded, there is a finite subset $F \subset M$ such that $d_H(D_0, D) < \delta$ where $D = \{a_n : n \in F\}$. Take any $m \in M$ with $m > \max F$ and find an η_m -chain of diameter $< \frac{\varepsilon_0}{16}$ connecting D_0 and D . Using this chain construct an η_m -chain connecting $a_m \in D_0$ with some point $a_n \in D$, $n \in F$, and lying in the $\frac{\varepsilon_0}{16}$ -ball around D_0 . This means that the diameter of the latter chain is $< \text{diam}D_0 + 2\frac{\varepsilon_0}{16} \leq \frac{\varepsilon_0}{4}$ and thus the set $\{n, m\}$ is white.

For the real number δ defined above find a number $n \in M \setminus F$ such that $\text{diam}A_n < \delta$. For this number n find a finite subset $E \subset M$ such that $d_H(D_0, D) < \min\{\eta_n, \delta/2\}$ where $D = \{a_i : i \in E\}$. By the choice of M for any numbers $i < j$ in E the points a_i, a_j are connected by an η_j -chain of diameter $< \varepsilon_0/4$. Let L be the maximal length of these chains.

Consider two countable uniformly discrete subsets of B :

$$D_x = D \cup \{x_{n,k} : k \in I_n\} \text{ and } D_y = D \cup \{y_{n,k} : k \in I_n\}.$$

It follows that $\max\{d_H(D_0, D), d_H(D_0, D_x), d_H(D_0, D_y)\} < \delta$ and consequently, D can be linked with D_x and D_y by η_n -chains in $B(D_0, \frac{\varepsilon_0}{16})$ with length $\leq l$ for some l . Take any number $k > 2(L + l)$ and using these chains construct an η_n -chain of length $\leq l$ connecting the point $x_{n,k}$ with some point $x \in D$ and an η_n -chain of length $\leq l$ connecting the point $y_{n,k}$ with some point $y \in D$. These chains lie in $B(D_0, \frac{\varepsilon_0}{16})$. Taking into account that the point $a_n \in D_0$ can be connected with the points $x, y \in D$ by η_n -chains of length $\leq L$ and diameter $< \varepsilon_0/4$ we conclude that the points $x_{n,k}$ and $y_{n,k}$ can be connected by an η_n -chain of length $2(L + l) < k$, which lies in $B(D_0, \varepsilon_0/4)$ and hence has diameter $< \text{diam}D_0 + 2\frac{\varepsilon_0}{4} \leq \frac{5}{8}\varepsilon_0 < \varepsilon_0$, which contradicts to the choice of the points $x_{n,k}, y_{n,k}$. \square

To prove the implication (8) \Rightarrow (4) of Theorem 1 we will exploit the following simple lemma allowing us to transform traces into paths in hyperspaces. Below by $\text{cl}_X(A)$ we denote the closure of a subset $A \subset X$.

Lemma 10. *Let X be a metric space and $f : Q \rightarrow 2^X$ be a uniformly continuous map defined on a countable dense subset $Q \supset \{0, a\}$ of $[0, a]$. Then the map $g : [0, a] \rightarrow 2^X$ defined by $g : t \mapsto \text{cl}_X(\cup\{f(q) : |q - t| \leq \text{dist}(t, \{0, a\})\})$ is a continuous path such that $g(0) = f(0)$, $g(a) = f(a)$ and $\omega_g \leq \omega_f$.*

5. PROOF OF THEOREM 3

Let (X, d) be a metric space and $\mathcal{H} \subset 2^X$ be an open subspace such that $\mathcal{D}_0(F) \subset \mathcal{H}$ for all $F \in \mathcal{H}$. The implication (2) \Rightarrow (1) follows from Proposition 1 while (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) are trivial. The implications (4) \Rightarrow (5) and (5) \Rightarrow (6) follow from Lemmas 9 and 2, respectively.

Now we verify the implication (6) \Rightarrow (2). To show that \mathcal{H} is locally path-connected, fix any element $F \in \mathcal{H}$ and $\varepsilon > 0$. Since the set \mathcal{H} is open in 2^X there is $r > 0$ such that

any $A \in 2^X$ with $d_H(A, F) < 2r$ belongs to \mathcal{H} . In particular, the closed r -neighborhood $\overline{B}(F, r) = \{x \in X : d(x, F) \leq r\}$ belongs to \mathcal{H} and thus X is uniformly locally trace equi-connected at $\overline{B}(F, r)$ by (6). Consequently, there are $\varepsilon_0 > 0$ and a continuity modulus ω such that for any points $x, y \in \overline{B}(F, r)$ with $d(x, y) < \varepsilon_0$ and any countable dense subset $Q \supset \{0, d(x, y)\}$ of $[0, d(x, y)]$ there is a uniformly continuous function $f : Q \rightarrow X$ with $f(0) = x$, $f(d(x, y)) = y$ and $\omega_f \leq \omega$. Without loss of generality we may assume that $\omega(\varepsilon_0) < \min\{\varepsilon, r\}$.

We claim that any element $A \in \mathcal{H}$ with $d_H(A, F) < \varepsilon_0$ can be linked with F by a path $g : [0, \varepsilon_0] \rightarrow \mathcal{H}$ lying in the ε -ball around F in 2^X . Pick any countable dense subset $Q \supset \{0, \varepsilon_0\}$ of $[0, \varepsilon_0]$. For any points $a \in A$, $b \in F$ find points $b_a \in F$, $a_b \in A$ with $d(a, b_a) < \varepsilon_0$ and $d(b, a_b) < \varepsilon_0$. For the obtained pairs (a, b_a) , (b, a_b) fix uniformly continuous maps $f_a : Q \rightarrow X$, $f_b : Q \rightarrow X$ such that $f_a(0) = a$, $f_a(\varepsilon_0) = b_a$, $f_b(0) = a_b$, $f_b(\varepsilon_0) = b$, and $\omega_{f_a} \leq \omega$, $\omega_{f_b} \leq \omega$. Define a trace $f : Q \rightarrow 2^X$ letting $f(q) = \text{cl}_X(\{f_a(q), f_b(q) : a \in A, b \in B\})$ for $q \in Q$. It can be shown that $f(0) = A$, $f(\varepsilon_0) = F$, and $\omega_f \leq \omega$. Using Lemma 10 produce a continuous path $g : [0, \varepsilon_0] \rightarrow 2^X$ defined by $g(t) = \text{cl}_X(\cup\{f(q) : |q - t| \leq \text{dist}(t, \{0, \varepsilon_0\})\})$ for $t \in [0, \varepsilon_0]$. For this path we get $g(0) = A$, $g(\varepsilon_0) = F$, and $\omega_g \leq \omega_f \leq \omega$, which implies that $g([0, \varepsilon_0])$ lies in the closed $\omega(\varepsilon_0)$ -ball around F in 2^X . Since $\omega(\varepsilon_0) < \min\{\varepsilon, r\}$ this ball lies in \mathcal{H} , which means that \mathcal{H} is locally path connected.

6. PROOF OF THEOREM 1

The implication (4) \Rightarrow (3) follows from Proposition 2 while (3) \Rightarrow (1) \Rightarrow (2) \Rightarrow (5) \Rightarrow (6) are trivial. The implication (6) \Rightarrow (7) is proven in Lemma 9 while (7) \Rightarrow (8) \Rightarrow (9) \Rightarrow (7) in Lemmas 2, 5 and 3, respectively. The implication (8) \Rightarrow (4) can be proved by analogy with the proof of (6) \Rightarrow (2) from Theorem 3. For a complete X the equivalence (8) \Leftrightarrow (10) is trivial.

The implication (11) \Rightarrow (3) is trivial. Assuming that X is uniformly locally compact we shall prove the implication (10) \Rightarrow (11). Assuming that X is uniformly locally path equi-connected, and applying Proposition 3, we conclude that the space $\mathcal{D}(X)$ is a uniform ANR. According to a result of K.Sakai [Sa], a metric space is uANR provided it contains a dense uANR. Consequently, each space $\mathcal{H} \subset 2^X$ containing $\mathcal{D}(X)$ is uANR.

Finally, assuming that X is chain equi-connected, we shall verify the implications (3) \Rightarrow (13) \Rightarrow (12) \Rightarrow (1). Assume that 2^X is a uANR. Since the space X is chain equi-connected we can apply Proposition 4.6 of [CK] to conclude that 2^X is chain connected (in should be mentioned that in [CK] chain equi-connected sets are called uniformly C -connected). Now Proposition 2 implies that 2^X is a uniform AR. This completes the proof of (3) \Rightarrow (13). Two other implications are trivial.

7. PROOF OF THEOREM 4

Assume that for each bounded subset B of X the space X is uniformly locally chain equi-connected at B and B lies in a bounded chain equi-connected subspace of X . By Corollary 1, the hyperspace $\text{Bd}(X)$ is an ANR and X is uniformly locally trace equi-connected at each bounded subset of X . In light of Proposition 1 to show that $\text{Bd}(X)$ is an AR, it suffices to verify that the space $\text{Bd}(X)$ is path connected. For this we shall show that each one-point subset $\{x_0\} \in 2^X$ can be linked with a non-empty closed bounded subset $A \in 2^X$ by a continuous path in $\text{Bd}(X)$. Our assumption implies that $\{x_0\} \cup A$ lies in a bounded chain equi-connected subspace $C \subset X$.

Since X is uniformly locally trace equi-connected at the bounded set C there is $\varepsilon_0 < 1$ and a continuity modulus ω such that any two points $x, y \in C$ with $d(x, y) < \varepsilon_0$ can be linked by a trace $f : \mathbb{Q} \cap [0, 1] \rightarrow X$ defined on the space of rational numbers of $[0, 1]$ such that $f(0) = x$, $f(1) = y$, and $\omega_f \leq \omega$.

Use the chain equi-connectedness of C to find $l \in \mathbb{N}$ such that any points $x, y \in C$ can be linked by an ε_0 -chain of length $\leq l$. Hence for each point $a \in A \subset C$ we can pick a map $f_a : \{0, \dots, l\} \rightarrow C$ such that $f_a(0) = x_0$, $f_a(l) = a$, and $d(f_a(i-1), f_a(i)) < \varepsilon_0$. By the choice of ε_0 and ω we can extend f_a to a uniformly continuous map $f_a : \mathbb{Q} \cap [0, l] \rightarrow X$ whose continuity modulus does not exceed $l\omega$. Then the function $f : \mathbb{Q} \cap [0, l] \rightarrow \text{Bd}(X)$ defined by $f : q \mapsto \text{cl}_X(\cup\{f_a(q) : a \in A\})$ is a trace in $\text{Bd}(X)$ linking $\{x_0\}$ with A . Finally, use Lemma 10

to transform f into a continuous path in $\text{Bd}(X)$ connecting $\{x_0\}$ and A . Therefore $\text{Bd}(X)$ is a path connected ANR. Applying Proposition 1 we conclude that $\text{Bd}(X)$ is an AR.

Now assume that $\text{Bd}(X)$ is an AR. By Corollary 1, X is uniformly locally chain equi-connected at each bounded subset B . Since $\text{Bd}(X)$ is path connected we can apply Proposition 4.7 of [CK] to conclude that each bounded subset of X lies in a bounded chain equi-connected subspace of X (in should be mentioned that in [CK] chain equi-connected sets are called uniformly C -connected).

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