# TOPOLOGICAL GROUPS AND CONVEX SETS HOMEOMORPHIC TO NON-SEPARABLE HILBERT SPACES

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Abstract. Let $X$ be a topological group or a convex set in a linear metric space. We prove that $X$ is homeomorphic to (a manifold modeled on) an infinite-dimensional Hilbert space if and only if $X$ is a compltetely metrizable absolute (neighborhood) retract with $\omega$-LFAP, the countable locally finite approximation property. The latter means that for any open cover $\mathcal{U}$ of $X$ there is a sequence of maps $(f_n : X \to X)_{n\in\omega}$ such that each $f_n$ is $\mathcal{U}$-near to the identity map of $X$ and the family $\{f_n(X)\}_{n\in\omega}$ is locally finite in $X$. Also we show that a metrizable space $X$ of density $\mathrm{dens}(X) < \mathfrak{d}$ is a Hilbert manifold if $X$ has $\omega$-LFAP and each closed subset $A \subset X$ of density $\mathrm{dens}(A) < \mathrm{dens}(X)$ is a $Z_\infty$-set in $X$.

## 1. Introduction

One of the most important achievements of the classical infinite-dimensional topology is

**Theorem 1** (Anderson-Kadec). *Each separable Fréchet (=locally convex complete linear metric) space is homeomorphic to a Hilbert space.*

The initial proof of this topological theorem was essentially geometric and used the Kadec's renorming technique. A simple topological proof was found in 80s by H.Toruńczyk who applied his elegant characterization of the Hilbert space topology [Tor] to generalize the Anderson-Kadec Theorem in three different directions. Firstly, he established the topological equivalence of any (not necessarily separable) Fréchet space to a Hilbert space.

**Theorem 2** (Toruńczyk). *Each Fréchet space is homeomorphic to a Hilbert space.*

Next, in their joint paper [DT] T. Dobrowolski and H. Toruńczyk generalized the Anderson-Kadec Theorem to topological groups and convex sets in linear metric spaces.

**Theorem 3** (Dobrowolski-Toruńczyk). *A Polish topological group $G$ is a Hilbert manifold if and only if $G$ is an absolute neighborhood retract.*

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**Theorem 4** (Dobrowolski-Toruńczyk)**.** *A convex subset $X$ of a complete linear metric space $L$ is homeomorphic to the separable Hilbert space $l_2$ provided $X$ is a Polish absolute retract and the closure of $X$ in $L$ is not locally compact.*

By an *absolute (neighborhood) retract* we understand a metrizable space $X$ which is a (neighborhood) retract in each metrizable space $M$ containing $X$ as a closed subspace.

Let us observe that the latter two theorems (unlike the former) say nothing about non-separable topological groups or convex sets. In this paper we shall try to fill this gap and shall address two natural (and still open) problems.

**Problem 1.** *Is each non-separable closed convex subset of a Fréchet space homeomorphic to a Hilbert space?*

**Problem 2.** *Let $G$ be a topological group whose underlying topological space a completely metrizable absolute neighborhood retract. Is $G$ a Hilbert manifold?*

By a *Hilbert manifold* we understand a paracompact space $X$ that admits a cover by open sets homeomorphic to open subsets of some Hilbert space. A topological space $X$ is *completely metrizable* if the topology of $X$ is generated by a complete metric.

The following theorem characterizes topological groups and convex sets homeomorphic to infinite-dimensional Hilbert spaces or Hilbert manifolds.

**Theorem 5.** *Let $X$ be either a topological group or a convex set in a linear metric space. The space $X$ is homeomorphic to an infinite-dimensional Hilbert space (to a Hilbert manifold) if and only if $X$ is a completely metrizable absolute (neighborhood) retract with $\omega$-LFAP.*

A topological space $X$ is said to have the *Countable Locally Finite Approximation Property* (briefly, $\omega$-LFAP) if for every open cover $\mathcal{U}$ there is a sequence of maps $(f_n : X \to X)_{n\in\omega}$ such that each $f_n$ is $\mathcal{U}$-near to the identity map $\mathrm{id} : X \to X$ and the family $\{f_n(X)\}_{n\in\omega}$ is locally finite in $X$.

We recall that two maps $f, g : Z \to X$ are *$\mathcal{U}$-near* if for every $z \in Z$ there is $U \in \mathcal{U}$ containing both the points $f(z)$ and $g(z)$.

Theorem 5 will be applied in [BSYZ] for recognizing the Hilbert space topology of connected components of the space $\mathrm{Conv}_H(X)$ of closed convex subsets of a Banach space $X$, endowed with the Hausdorff metric.

It seems that $\omega$-LFAP can be also applied to recognize the Hilbert space topology in spaces without additional algebraic structure.

**Conjecture 1.** *A completely metrizable absolute neighborhood retract $X$ with $\omega$-LFAP is a Hilbert manifold if and only if each closed subspace $A \subset X$ of density* $\mathrm{dens}(A) < \mathrm{dens}(X)$ *is a $Z_\infty$-set in $X$.*

We recall that a closed subset $A \subset X$ is called a $Z_\infty$-*set* in $X$ if each map $f : I^n \to X$ from a finite-dimensional cube can be uniformly approximated by maps into $X \setminus A$.

We shall confirm Conjecture 1 for spaces $X$ with density $\mathrm{dens}(X) < \mathfrak{d}$. Here the cardinal $\mathfrak{d}$ is well-known in Set Theory as the *dominating number*. It is equal to the cofinality of the partially ordered set $\omega^\omega$ (see [vD], [Va]) but can also be equivalently defined as the smallest size $|\mathcal{C}|$ of a cover $\mathcal{C}$ of the Hilbert space $l_2$ by compact subsets. Under Martin Axiom, $\mathfrak{d}$ equals the cardinality of continuum $\mathfrak{c}$ but there are models of ZFC with $\mathfrak{d} < \mathfrak{c}$, see [vD].

**Theorem 6.** *A completely metrizable connected absolute neighborhood retract $X$ of density $\mathrm{dens}(X) < \mathfrak{d}$ is an infinite-dimensional Hilbert manifold if and only if $X$ has $\omega$-LFAP and each closed subspace $A \subset X$ of density $\mathrm{dens}(A) < \mathrm{dens}(X)$ is a $Z_\infty$-set in $X$.*

## 2. Some conventions and notations

By $I$ we shall denote the unit interval $[0, 1]$. The cardinality of a set $X$ is denoted by $|X|$. Cardinals are identified with the sets of ordinals of smaller size and are endowed with the discrete topology.

For a metric space $X$ and two points $x, y \in X$ by $\mathrm{dist}(x, y)$ we shall denote the distance between $x, y$. More generally, for two subsets $A, B \subset X$ we put $\mathrm{dist}(A, B) = \inf\{\mathrm{dist}(a, b) : a \in A,\ b \in B\}$. For a point $x$ in a metric space $X$ and $\varepsilon > 0$ by $B(x, \varepsilon) = \{y \in X : \mathrm{dist}(x, y) < \varepsilon\}$ we denote the open $\varepsilon$-ball centered at $x$.

For a subset $A$ of a space $X$ and a cover $\mathcal{U}$ of $X$ we put $\mathcal{S}t(A,\mathcal{U}) = \cup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$ and $\mathcal{S}t(\mathcal{U}) = \{\mathcal{S}t(U,\mathcal{U}) : U \in \mathcal{U}\}$. For two families $\mathcal{U}, \mathcal{V}$ of subsets of a space $X$ we write $\mathcal{U} \prec \mathcal{V}$ and say that $\mathcal{U}$ is inscribed in $\mathcal{V}$ (or else $\mathcal{U}$ refines $\mathcal{V}$) if each set $U \in \mathcal{U}$ lies in some set $V \in \mathcal{V}$.

An indexed family $\{F_\alpha\}_{\alpha\in A}$ of subsets of a space $X$ is called locally finite (discrete) in $X$ if for each point $x \in X$ there are a neighborhood $Ox \subset X$ and a finite subset $B \subset A$ (with $|B| \le 1$) such that $Ox \cap F_\alpha = \emptyset$ for all $\alpha \in A \setminus B$.

We shall often use the following elementary lemma whose proof can be found in [Ban, Lemma 1].

**Lemma 1.** *For any locally finite collection $\{F_\alpha\}_{\alpha\in A}$ of subsets of a paracompact space $X$, there is an open cover $\mathcal{U}$ of $X$ such that the family $\big\{\mathcal{S}t(F_\alpha,\mathcal{U})\big\}_{\alpha\in A}$ is locally finite.*

## 3. $\kappa$-Discrete $m$-Cells Property and Toruńczyk's characterization of the Hilbert space topology

In fact, $\omega$-LFAP is one of two principal ingredients composing the famous Toruńczyk's characterization of the Hilbert space topology. The other one is the $\kappa$-*discrete* $m$-*cells property* defined as follows:

A topological space $X$ is said to satisfy the *$\kappa$-discrete $m$-cells property* for cardinals $\kappa$ and $m$ if for every map $f:\kappa\times I^m\to X$ and every open cover $\mathcal{U}$ of $X$ there is a map $g:\kappa\times I^m\to X$ such that $g$ is $\mathcal{U}$-near to $f$ and the family $\big\{g(\{\alpha\}\times I^m)\big\}_{\alpha\in\kappa}$ is discrete in $X$ (according to our convention, the cardinal $\kappa$ is endowed with the discrete topology).

Theorems 5 and 6 will be proved with help of the famous characterization of Hilbert manifolds due to H. Toruńczyk [Tor]:

**Theorem 7** (Toruńczyk)**.** *A connected topological space $X$ of density $\kappa$ is a Hilbert manifold if and only if*

(1) *$X$ is a completely-metrizable absolute neighborhood retract;*
(2) *$X$ has $\omega$-LFAP;*
(3) *$X$ has the $\kappa$-discrete $m$-cells property for every $m<\omega$.*

Therefore, for completely-metrizable ANR's with $\omega$-LFAP, recognizing the Hilbert space topology reduces to establishing the $\kappa$-discrete $m$-cells property. In this respect the following lemma established in [Ban, Lemma 4] can be helpful.

**Lemma 2.** *A topological space $X$ has the $\kappa$-discrete $m$-cells property for an infinite cardinal $\kappa$ if and only if for every open cover $\mathcal{U}$ of $X$ and a map $f:\kappa\times I^m\to X$ there is a map $g:\kappa\times I^m\to X$ such that $g$ is $\mathcal{U}$-near to $f$ and the family $\big\{g(\{\alpha\}\times I^m)\big\}_{\alpha<\kappa}$ is locally finite in $X$.*

For spaces with $\omega$-LFAP the verification of the $\kappa$-discrete $m$-cells property can be reduced to checking this property for cardinals with uncountable cofinality.

**Lemma 3.** *A paracompact space $X$ with $\omega$-LFAP has $\kappa$-discrete $m$-cells property for a cardinal $\kappa$ if and only if $X$ has the $\lambda$-discrete $m$-cells property for all cardinals $\lambda\le\kappa$ of uncountable cofinality.*

*Proof.* The "only if" part is trivial. To prove the "if" part, assume that $X$ has the $\lambda$-discrete $m$-cells property for all cardinals $\lambda\le\kappa$ of uncountable cofinality and consider three possible cases.

1. If $\kappa\le\omega$ then the $\kappa$-discrete $m$-cells property of $X$ follows from $\omega$-LFAP and Lemma 2.

2. If $\kappa$ has uncountable cofinality, then there is nothing to prove.

3. Finally assume that $\kappa$ is an uncountable cardinal with countable cofinality. According to Lemma 2 the $\kappa$-discrete $m$-cells property of $X$ will follow as soon as given an open cover $\mathcal{U}$ of $X$ and a family of maps $\big\{f_\alpha:I^m\to X\big\}_{\alpha\in\kappa}$ we shall construct a family of maps $\big\{g_\alpha:I^m\to X\big\}_{\alpha\in\kappa}$ such that each $g_\alpha$ is $\mathcal{U}$-near to $f_\alpha$ and the family $\big\{g_\alpha(I^m)\big\}_{\alpha\in\kappa}$ is locally finite in $X$.

By the paracompactness of $X$, find an open cover $\mathcal{V}$ of $X$ with $\mathcal{S}t(\mathcal{V})\prec\mathcal{U}$. Using $\omega$-LFAP, fix a sequence of maps $\{h_n:X\to X\}_{n\in\omega}$ such that each $h_n$ is $\mathcal{V}$-near to the identity map of $X$ and the family $\big\{h_n(X)\big\}_{n\in\omega}$ is locally finite. By Lemma 1, there is an open cover $\mathcal{W} \prec \mathcal{V}$ such that the family $\{\mathcal{S}t(h_n(X), \mathcal{W})\}_{n\in\omega}$ is locally finite.

The cardinal $\kappa$ of countable cofinality can be written as the countable union $\kappa = \bigcup_{n\in\omega} \kappa_n$ of pairwise disjoint subsets $\kappa_n \subset \kappa$ of size $|\kappa_n| < \kappa$, where each cardinal $|\kappa_n|$ has uncountable cofinality (for example, is a successor cardinal). For every $n \in \omega$ use the $\kappa_n$-discrete $m$-cells property of $X$, to construct a family of maps $\{g_\alpha : I^m \to X\}_{\alpha\in\kappa_n}$ such that each map $g_\alpha$ is $\mathcal{W}$-near to $h_n \circ f_\alpha$ and the family $\{g_\alpha(I^m)\}_{\alpha\in\kappa_n}$ is discrete in $X$. Taking into account that $\mathcal{W} \prec \mathcal{V}$, $\mathcal{S}t(\mathcal{V}) \prec \mathcal{U}$, and $h_n \circ f_\alpha$ is $\mathcal{V}$-near to $f_\alpha$, we conclude that $g_\alpha$ is $\mathcal{U}$-near to $f_\alpha$.

Unifying the families $\{g_\alpha\}_{\alpha\in\kappa_n}$, $n \in \omega$, we shall obtain a desired family of maps $\{g_\alpha\}_{\alpha\in\kappa}$ such that each $g_\alpha$ is $\mathcal{U}$-near to $f_\alpha$ and the family $\{g_\alpha(I^m)\}_{\alpha\in\kappa}$ is locally finite. □

## 4. The $\kappa$-discrete $m$-cells property and $Z_m$-sets

In this section we reveal the interplay betwen the $\kappa$-discrete $m$-cells property and $Z_m$-sets.

A closed subset $A$ of a topological space $X$ will be called a *$Z_m$-set* in $X$ if for each map $f : I^m \to X$ and an open cover $\mathcal{U}$ of $X$ there is a map $g : I^m \to X$ such that $g$ is $\mathcal{U}$-near to $f$ and $g(I^m) \cap A = \emptyset$. Let us observe that a closed subset $A$ of a metrizable space $X$ is a $Z_\infty$-set if and only if $A$ is a $Z_m$-set for all (finite) cardinals $m$.

By the *Lindelöf number* $l(X)$ of a topological space $X$ we understand the smallest cardinal $\kappa$ such that each open cover of $X$ has a subcover of size $\leq \kappa$. It is known that the Lindelöf number of a metrizable space $X$ is equal to the density and weight of $X$.

**Lemma 4.** *If a topological space $X$ has the $\kappa$-discrete $m$-cells property for some cardinals $m$ and $\kappa$, then each closed subset $A \subset X$ with Lindelöf number $l(A) < \kappa$ is a $Z_m$-set in $X$.*

*Proof.* Let $A$ be a closed subspace of $X$ with Lindelöf number $l(A) < \kappa$. To show that $A$ is a $Z_m$-set in $X$, fix a map $f : I^m \to X$ and an open cover $\mathcal{U}$ of $X$. Applying the $\kappa$-discrete $m$-cells property of $X$, find a family of maps $\{f_\alpha : I^m \to X\}_{\alpha\in\kappa}$ such that each $f_\alpha$ is $\mathcal{U}$-near to $f$ and the family $\{f_\alpha(I^m)\}_{\alpha\in\kappa}$ is discrete in $X$. The latter means that for every $x \in X$ we can pick a neighborhood $Ox$ and an ordinal $\alpha(x) \in \kappa$ such that $Ox \cap f_\alpha(I^m) = \emptyset$ for all $\alpha \neq \alpha(x)$. Since $l(A) < \kappa$, the open cover $\{Ox : x \in A\}$ of $A$ has a subcover of size $< \kappa$. Consequently, we can find a subset $B \subset A$ of size $|B| < \kappa$ such that $\{Ox : x \in B\}$ is a cover of $A$. Take any ordinal $\alpha \in \kappa \setminus \{\alpha(x) : x \in B\}$ and note that $f_\alpha(I^m) \cap A \subset f_\alpha(I^m) \cap \bigcup_{x\in B} Ox = \emptyset$. Since $f_\alpha$ is $\mathcal{U}$-near to $f$, this witnesses that $A$ is a $Z_m$-set in $X$. □

For cardinals $\kappa < \mathfrak{d}$ this lemma can be partly reversed. The following lemma combined with the Toruńczyk's Theorem 7 implies Theorem 6.

**Lemma 5.** *Let $\kappa < \mathfrak{d}$ and $m \le \omega$ be two cardinals. A metrizable space $X$ with $\omega$-LFAP has the $\kappa$-discrete $m$-cells property if and only if each closed subset $A \subset X$ with* $\operatorname{dens}(A) < \kappa$ *is a $Z_m$-set in $X$.*

*Proof.* The "only if" part follows from Lemma 2. To prove the "if" part, assume that each closed subset $A \subset X$ with $\operatorname{dens}(A) < \kappa$ is a $Z_m$-set in $X$. Fix a continuous metric $\rho \le 1$ on $X$. To establish the $\kappa$-discrete $m$-cells property, fix an open cover $\mathcal{U}$ of $X$ and a family of maps $\{f_\alpha : I^m \to X\}_{\alpha\in\kappa}$. If $\kappa \le \omega$, then the $\kappa$-discrete $m$-cells property follows from $\omega$-LFAP and Lemma 2. So we assume that $\kappa$ is uncountable.

By the paracompactness of $X$ there is an open cover $\mathcal{V}$ of $X$ with $\mathcal{S}t(\mathcal{V}) \prec \mathcal{U}$. Using $\omega$-LFAP, fix a sequence of maps $\{g_n : X \to X\}_{n\in\omega}$ such that each $g_n$ is $\mathcal{V}$-near to the identity map $\mathrm{id} : X \to X$ and the family $\big\{g_n(X)\big\}_{n\in\omega}$ is locally finite in $X$. By Lemma 1, there is an open cover $\mathcal{W} \prec \mathcal{V}$ of $X$ such that the family $\big\{\mathcal{S}t(g_n(X),\mathcal{W})\big\}_{n\in\omega}$ is locally finite in $X$.

The product $\kappa\times\omega$ carries a natural lexicographic ordering $<$ turning it into a well-ordered set isomorphic to $\kappa$.

Using the fact that each closed subset $A \subset X$ with $\operatorname{dens}(A) < \kappa$ is a $Z_m$-set, by transfinite induction we can construct a transfinite sequence of maps $(h_{(\alpha,n)} : I^m \to X)_{(\alpha,n)\in\kappa\times\omega}$ such that for every $(\alpha,n)\in\kappa\times\omega$

- $h_{(\alpha,n)}$ is $\mathcal{W}$-near to $g_n\circ f_\alpha$;
- $h_{(\alpha,n)}(I^m)$ misses the closure $A_{(\alpha,n)}$ of the union $\bigcup_{(\beta,k)<(\alpha,n)} h_{(\beta,k)}(I^m)$ (which has density $<\kappa$ and hence is a $Z_m$-set in $X$).

After completing the transfinite construction, for each $(\alpha,n)\in\kappa\times\omega$ consider the positive real number

$$\xi(\alpha,n) = \operatorname{dist}(h_{(\alpha,n)}(I^m), A_{(\alpha,n)}) = \inf\{\rho(x,y) : x\in h_{(\alpha,n)}(I^m),\ y\in A_{(\alpha,n)}\}.$$

For every $\alpha\in\kappa$ the function $\xi_\alpha : \omega\to(0,1]$, $\xi_\alpha : n\mapsto\xi(\alpha,n)$, determines a compact subset $K_\alpha = \prod_{n\in\omega}[\xi_\alpha(n),1]$ in $(0,1]^\omega$. Now the strict inequality $\kappa<\mathfrak{d}$ and the definition of the cardinal $\mathfrak{d}$ (as the compact covering number of $(0,1]^\omega \cong l^2$) imply that the family of compacta $\{K_\alpha : \alpha\in\kappa\}$ does not cover $(0,1]^\omega$, which implies the existence of a function $\xi\in(0,1]^\omega\setminus\bigcup_{\alpha\in\kappa}K_\alpha$.

For every ordinal $\alpha\in\kappa$ the non-inclusion $\xi\notin K_\alpha$ implies the existence of a number $n_\alpha\in\omega$ such that $\xi(n_\alpha)<\xi_\alpha(n_\alpha)$. Given such a number $n_\alpha$, consider the map $f'_\alpha = h_{(\alpha,n_\alpha)} : I^m\to X$, which is $\mathcal{S}t(\mathcal{V})$-near to $f_\alpha$ because $h_{(\alpha,n_\alpha)}$ is $\mathcal{V}$-near to $g_{n_\alpha}\circ f_\alpha$ and $g_{n_\alpha}\circ f_\alpha$ is $\mathcal{V}$-near to $f_\alpha$. Since $\mathcal{S}t(\mathcal{V})\prec\mathcal{U}$, $f'_\alpha$ is $\mathcal{U}$-near to $f_\alpha$.

It remains to check that the family $\{f'_\alpha(I^m)\}_{\alpha\in\kappa}$ is locally finite in $X$. For every $n\in\omega$ let $\kappa_n = \{\alpha\in\kappa : n_\alpha = n\}$. It follows from the definition of the numbers $\xi(\alpha,n)$ that for any distinct ordinals $\alpha,\beta$ in $\kappa_n$ we get $\operatorname{dist}(f'_\alpha(I^m), f'_\beta(I^m)) \ge \xi(n) > 0$, which means that the family $\big\{f'_\alpha(I^m)\big\}_{\alpha\in\kappa_n}$ is discrete in $X$. Since $\bigcup_{\alpha\in\kappa_n} f'_\alpha(I^m) \subset \mathcal{S}t(g_n(X),\mathcal{W})$ for all $n\in\omega$, the local finiteness of the family $\big\{\mathcal{S}t(g_n(X),\mathcal{W})\big\}_{n\in\omega}$ implies the local finiteness of the family $\big\{f_\alpha(I^m)\big\}_{\alpha\in\kappa} = \bigcup_{n\in\omega}\{f'_\alpha(I^m)\}_{\alpha\in\kappa_n}$. □

## 5. $\kappa$-Discrete $m$-Cells Property in Metric Spaces

In this section we introduce a metric counterpart of the $\kappa$-discrete $m$-cells property called the $\kappa$-separated $m$-cells property and shall detect this property in pointed metric spaces with a pseudo-translation.

We shall say that a metric space $(X,\rho)$ has the *$\kappa$-separated $m$-cells property* if for every $\varepsilon>0$ there is $\delta>0$ such that for every map $f:\kappa\times I^m\to X$ there is a map $g:\kappa\times I^m\to X$ that is $\varepsilon$-homotopic to $f$ and such that $\mathrm{dist}\big(g(\{\alpha\}\times I^m),g(\{\beta\}\times I^m)\big)\ge\delta$ for all ordinals $\alpha<\beta<\kappa$.

We recall that two maps $f,g:Z\to X$ to a metric space $(X,\rho)$ are *$\varepsilon$-homotopic* if there is a homotopy $h:Z\times[0,1]\to X$ such that $h(z,0)=f(z)$, $h(z,1)=g(z)$ and $\mathrm{diam}_\rho h(\{z\}\times I)<\varepsilon$ for all $z\in Z$.

By transfinite induction the following characterization of the $\kappa$-separated $m$-cells property can be established.

**Lemma 6.** *Let $m\le\omega$ and $\kappa$ be an uncountable cardinal. A metric space $(X,\rho)$ has the $\kappa$-separated $m$-cells property if and only if for every $\varepsilon>0$ there is $\delta>0$ such that for every subset $A\subset X$ with density $\mathrm{dens}(A)<\kappa$, and every map $f:I^m\to X$ there is a map $g:I^m\to X$ that is $\varepsilon$-homotopic to $f$ and has $\mathrm{dist}(g(I^m),A)>\delta$.*

The following important lemma describing the interplay between the $\kappa$-discrete and $\kappa$-separated $m$-cells properties will be proved by the argument of the proof of Lemma 1 in [DT].

**Lemma 7.** *Each metric space $X$ with the $\kappa$-separated $m$-cells property has the $\kappa$-discrete $m$-cells property.*

*Proof.* Given a map $f:\kappa\times I^m\to X$ and an open cover $\mathcal{U}$ of the metric space $X$, find a non-expanding map $\varepsilon:X\to(0,1)$ such that for any $x\in X$ the open $\varepsilon(x)$-ball $B(x,\varepsilon(x))$ centered at $x$ lies in some element $U\in\mathcal{U}$ of the cover $\mathcal{U}$ (the non-expanding property of $\varepsilon$ means that $|\varepsilon(x)-\varepsilon(y)|\le\mathrm{dist}(x,y)$ for all $x,y\in X$).

Let $D=\kappa\times I^m$ and $I^m_\alpha=\{\alpha\}\times I^m$ for $\alpha<\kappa$. For every $n\in\mathbb{Z}$ consider the closed subset

$$D_n=\{x\in D:\varepsilon\circ f(x)\ge 4^{-n}\}$$

of $D=\kappa\times I^m$ and note that $D_n=\emptyset$ for $n\le 0$.

By induction we shall construct sequences of maps $\{f_n:D\to X\}_{n\in\omega}$ and positive reals $\{\varepsilon_n\}_{n\in\omega}$ such that for every $n>0$ the following conditions are satisfied:

($1_n$) $f_n=f_{n-1}$ on $D_{n-2}\cup(D\setminus D_{n+1})$;
($2_n$) $\mathrm{dist}\big(f_n(D_n\cap I^m_\alpha),f_n(I^m_\beta)\big)>4\varepsilon_n$ for all $\beta<\alpha<\kappa$;
($3_n$) $\mathrm{dist}(f_n(x),f_{n-1}(x))<\varepsilon_{n-1}$ for $x\in D$;
($4_n$) $\varepsilon_n<\frac14\varepsilon_{n-1}$.

Put $\varepsilon_0=\frac{3}{16}$, $f_0=f$, and suppose that $f_{n-1}$ and $\varepsilon_{n-1}$ are known. By Lemma 6, there is a positive real number $\varepsilon_n<\frac14\varepsilon_{n-1}$ such that for every subset $A\subset X$ with density $\mathrm{dens}(A)<\kappa$, and every map $g:I^m\to X$ there is a map $g' : I^m \to X$ that is $\varepsilon_{n-1}$-homotopic to $g$ and has $\mathrm{dist}(g'(I^m), A) > 6\varepsilon_n$.

By transfinite induction, for every $\alpha < \kappa$ we shall construct a map $g_\alpha : I^m_\alpha \to X$ such that

$(a_\alpha)$ $g_\alpha$ is $\varepsilon_{n-1}$-homotopic to $f_{n-1}|I^m_\alpha$;
$(b_\alpha)$ $g_\alpha$ coincides with $f_{n-1}$ on the set $I^m_\alpha \cap (D_{n-2} \cup D \setminus D_{n+1})$;
$(c_\alpha)$ $\mathrm{dist}(g_\alpha(I^m_\alpha \cap D_n), A_\alpha) > 4\varepsilon_n$ where $A_\alpha = \bigcup_{\beta<\alpha} g_\beta(I^m_\beta)$.

To start the inductive construction, put $g_0 = f_{n-1}$. Next, assume that for some $\alpha < \kappa$ the maps $g_\beta$, $\beta < \alpha$, have been constructed. The set $A_\alpha = \bigcup_{\beta<\alpha} g_\beta(I^m_\beta)$ has density $< \kappa$. So, the choice of $\varepsilon_n$ guarantees the existence of a map $h_\alpha : I^m_\alpha \to X$ that is $\varepsilon_{n-1}$-homotopic to $f_{n-1}|I^m_\alpha$ and has $\mathrm{dist}(h_\alpha(I^m_\alpha), A_\alpha) > 6\varepsilon_n$. Taking into account that $h_\alpha$ is $\varepsilon_{n-1}$-homotopic to $f_{n-1}|I^m_\alpha$, we can construct a map $g_\alpha : I^m_\alpha \to X$ coinciding with $h_\alpha$ on the set $I^m_\alpha \cap (D_n \setminus D_{n-1})$ and satisfying the conditions $(a_\alpha)$ and $(b_\alpha)$. We claim that the map $g_\alpha$ satisfies the condition $(c_\alpha)$. We need to check that $\mathrm{dist}(g_\alpha(x), A_\alpha) > 4\varepsilon_n$ for every $x \in I^m_\alpha \cap D_n$. If $x \in D_n \setminus D_{n-1}$, then $g_\alpha(x) = h_\alpha(x)$ and hence

$$\mathrm{dist}(g_\alpha(x), A_\alpha) = \mathrm{dist}(h_\alpha(x), A_\alpha) > 6\varepsilon_n > 4\varepsilon_n$$

by the choice of $h_\alpha$. If $x \in D_{n-1}$, then by the condition $(2_{n-1})$ we have $\mathrm{dist}(f_{n-1}(x), B_\alpha) > 4\varepsilon_{n-1}$, where $B_\alpha = \bigcup_{\beta<\alpha} f_{n-1}(I^m_\beta)$. Taking into account that $g_\beta$ is $\varepsilon_{n-1}$-homotopic to $f_{n-1}|I^m_\beta$ for $\beta \le \alpha$, we conclude that

$$\mathrm{dist}(g_\alpha(x), A_\alpha) \ge 4\varepsilon_{n-1} - 2\varepsilon_{n-1} = 2\varepsilon_{n-1} > 4\varepsilon_n.$$

This completes the inductive construction of the maps $g_\alpha$, $\alpha < \kappa$, after which we can define a map $f_n : D \to X$ letting $f_n|I^m_\alpha = g_\alpha$ for $\alpha < \kappa$. Now we see that the conditions $(a_\alpha)$–$(c_\alpha)$, $\alpha < \kappa$, imply the conditions $(1_n)$–$(3_n)$. This completes the inductive construction of the map $f_n$.

We claim that the map $f_\infty = \lim_{n\to\infty} f_n : D \to X$ witnesses the $\kappa$-discrete $m$-cells property of $X$. The conditions $(1_n)$, $(3_n)$, $(4_n)$, $n \in \omega$, imply that $f_\infty$ is a well-defined continuous map. We claim that $\mathrm{dist}(f_\infty(x), f(x)) < \varepsilon \circ f(x)$ for every $x \in D$.

Given any $x \in D$ find $n \in \omega$ with $x \in D_n \setminus D_{n-1}$. Then $4^{-n} \le \varepsilon \circ f(x) < 4^{n-1}$. The conditions $(1_i)$, $i \le n-2$, imply that $f_{n-2}(x) = f_0(x) = f(x)$ and consequently,

$$\mathrm{dist}(f_\infty(x), f(x)) = \mathrm{dist}(f_\infty(x), f_{n-2}(x)) \le \sum_{i=n-2}^{\infty} \mathrm{dist}(f_{i+1}(x), f_i(x)) <$$

$$< \sum_{i=n-2}^{\infty} \varepsilon_i \le \sum_{i=n-2}^{\infty} \frac{1}{4^i}\varepsilon_0 = \frac{4}{3}\frac{1}{4^{n-2}}\frac{3}{16} = 4^{-n} \le \varepsilon \circ f(x).$$

Finally, we check that the family $(f_\infty(I^m_\alpha))_{\alpha\in\kappa}$ is discrete in $X$. Take any point $x \in X$ and find $n \in \omega$ with $4^{-n} \le \varepsilon(x) < 4^{-n-1}$. The discreteness of the family $(f_\infty(I^m_\alpha))_{\alpha\in\kappa}$ will follow as soon as we prove that the open

$\varepsilon_{n+2}$-ball $B(x,\varepsilon_{n+2})$ centered at $x$ meets at most one set $f_\infty(I^m_\alpha)$, $\alpha\in\kappa$. Assume conversely that $B(x,\varepsilon_{n+2})$ meets two sets $f_\infty(I^m_\alpha)$ and $f_\infty(I^m_\beta)$ for some $\alpha<\beta<\kappa$. Pick points $z_\alpha\in I^m_\alpha$ and $z_\beta\in I^m_\beta$ with $\{f_\infty(z_\alpha),f_\infty(z_\beta)\}\subset B(x,\varepsilon_{n+2})$ and observe that

$$\mathrm{dist}(f_\infty(z_\alpha),f_\infty(z_\beta))<2\varepsilon_{n+2}.$$

We claim that $z_\alpha\in D_{n+1}$. Assuming the converse, we would get $f(z_\alpha)<4^{-n-1}$. The non-expanding property of the map $\varepsilon$ implies

$$|\varepsilon\circ f_\infty(z_\alpha)-\varepsilon\circ f(z_\alpha)|\le \mathrm{dist}(f_\infty(z_\alpha),f(z_\alpha))<\varepsilon\circ f(z_\alpha)$$

and thus

$$\varepsilon\circ f_\infty(z_\alpha)<2\,\varepsilon\circ f(z_\alpha)<2\cdot 4^{-n-1}\le\frac{1}{2}\varepsilon(x).$$

Consequently,

$$\frac{1}{2}4^{-n}\le\frac{1}{2}\varepsilon(x)\le|\varepsilon(x)-\varepsilon\circ f_\infty(z_\alpha)|\le\mathrm{dist}(x,f_\infty(z_\alpha))<\varepsilon_{n+2}$$

which contradicts the inequality $\varepsilon_{n+2}<4^{-n-2}\varepsilon_0<\frac{1}{2}4^{-n}$ ensured by $(4_i)$, $i\le n+2$. Thus $z_\alpha\in D_{n+1}$. By the same reason, $z_\beta\in D_{n+1}$.

It follows from $(1_i)$, $i\ge n+3$, that $f_\infty(z_\alpha)=f_{n+2}(z_\alpha)$ and $f_\infty(z_\beta)=f_{n+2}(z_\beta)$. On the other hand, the condition $(2_{n+2})$ guarantees that

$$4\varepsilon_{n+2}<\mathrm{dist}(f_{n+2}(z_\beta),f_{n+2}(z_\alpha))=\mathrm{dist}(f_\infty(z_\beta),f_\infty(z_\alpha))<2\varepsilon_{n+2},$$

and this is a desired contradiction. □

## 6. Metric spaces with a pseudo-translation

In this section we establish the $\kappa$-separated $m$-cells property in pointed metric spaces with a pseudo-translation.

A metric space $X$ with a distinguished point $\theta$ is defined to have a *pseudo-translation* if there is a continuous map $\mu:X\times X\to X$ such that $\mu(x,\theta)=x$ and $\mathrm{dist}(\mu(x,y),\mu(x,z))=\mathrm{dist}(y,z)$ for all $x,y,z\in X$. The point $\theta$ will be called the *origin* of $X$ and the map $\mu$ will be refered to as a *pseudo-translation* of $(X,\theta)$.

A typical example of a pseudo-translation is the group multiplication on a topological group $G$ endowed with a left-invariant metric. More generally, each submonoid $X\subset G$ has a pseudo-translation. A subset $X\subset G$ is called a *submonoid* of $G$ if $X$ contains the neutral element of $G$ and $xy\in X$ for any points $x,y\in X$. The neutral element is the distinguished point of $X$.

A subset $S$ of a metric space $X$ is called *separated* if there is $\delta>0$ such that $\mathrm{dist}(x,y)\ge\delta$ for every distinct points $x,y\in S$.

**Lemma 8.** *Let $m\le\omega$, and $\kappa$ be an uncountable cardinal. Let $X$ be a pointed metric space with a pseudo-translation and assume that $X$ is locally path-connected at the origin $\theta$ of $X$. The space $X$ has the $\kappa$-separated $m$-cells property if and only if each neighborhood of $\theta$ contains a separated subset of size $\kappa$.*

*Proof.* To prove the "only if" part, take any neighborhood $U \subset X$ of the origin $\theta$ and find $\varepsilon > 0$ such that $U$ contains the $\varepsilon$-ball centered at $\theta$. Assuming that $X$ has the $\kappa$-separated $m$-cells property, for the constant map $f : \kappa \times I^m \to \{\theta\}$ find a map $g : \kappa \times I^m \to X$ such that $g$ is $\varepsilon$-homotopic to $f$ and

$$\delta = \inf\{\mathrm{dist}(g(\{\alpha\} \times I^m), g(\{\beta\} \times I^m)) : \alpha < \beta < \kappa\} > 0.$$

Then for any point $z \in I^m$ the set $S = \{g(\alpha, z) : \alpha < \kappa\}$ is $\delta$-separated, has size $\kappa$, and lies in $B(\theta, \varepsilon) \subset U$.

To prove the "if" part, assume that each neighborhood of $\theta$ contains a separated subset of size $\kappa$. Let $\mu : X \times X \to X$ be a pseudo-translation on the pointed space $(X, \theta)$. According to Lemma 6 the $\kappa$-separated $m$-cells property of $X$ will follow as soon as given $\varepsilon > 0$ we find $\delta > 0$ such that for every subset $A \subset X$ with $\mathrm{dens}(A) < \kappa$ and every $f : I^m \to X$ there is a map $g : I^m \to X$ which is $\varepsilon$-homotopic to $f$ and such that $\mathrm{dist}(g(I^m), A) \ge \delta$.

Since $X$ is locally path-connected at $\theta$, there is $\delta_1 > 0$ such that each point $y \in B(\theta, \delta_1)$ can be linked with $\theta$ by a path of diameter $< \varepsilon$. By our assumption, the $\delta_1$-ball $B(\theta, \delta_1)$ contains a separated subset $S \subset B(\theta, \delta_1)$ of size $|S| = \kappa$. Since $S$ is separated, the number

$$\delta = \frac{1}{2} \inf\{\rho(y, z) : y, z \in S,\ y \neq z\}$$

is strictly positive.

We claim that the number $\delta$ satisfies our requirements. Indeed, fix a subset $A \subset X$ with $\mathrm{dens}(A) < \kappa$. We claim that there is a point $z \in S$ such that $\mathrm{dist}(\mu(f(I^m) \times \{z\}), A) \ge \delta$. Assuming the converse, for every $z \in S$ we could find points $q_z \in f(I^m)$ and $a_z \in A$ with $\rho(\mu(q_z, z), a_z) < \delta$. We may select $(q_z, a_z)$ to belong to some fixed dense subset $Q \subset f(I^m) \times A$ having size $|Q| \le \mathrm{dens}(f(I^m) \times A) \le \max\{\omega, \mathrm{dens}(A)\} < \kappa$. The strict inequality $|Q| < \kappa$ implies the existence of two distinct points $y, z \in S$ with $(q_z, a_z) = (q_y, a_y)$. Let $x = q_z = q_y$ and observe that

$$\begin{aligned} 2\delta \le& \mathrm{dist}(y, z) = \mathrm{dist}(\mu(x, y), \mu(x, z)) \le \\ \le& \mathrm{dist}(\mu(q_y, y), a_y) + \mathrm{dist}(\mu(q_z, z), a_z) < 2\delta, \end{aligned}$$

which is a contradiction, proving the existence of a point $z \in S$ with $\mathrm{dist}(\mu(f(I^m) \times \{z\}), A) \ge \delta$. Define a map $g : I^m \to X$ letting $g(x) = \mu(f(x), z)$ for $x \in I^m$. It follows that $\mathrm{dist}(g(I^m), A) \ge \delta$.

By the choice of $\delta_1$ the point $z \in S \subset B(\theta, \delta_1)$ can be linked with $\theta$ by a path $\gamma : [0, 1] \to X$ with $\gamma(0) = \theta$, $\gamma(1) = z$ and $\mathrm{diam}(\gamma[0, 1]) < \varepsilon$. This path allows us to define an $\varepsilon$-homotopy

$$h : I^m \times [0, 1] \to X,\ h : (x, t) \mapsto \mu(f(x), \gamma(t))$$

linking the maps $f$ and $g = h_1$. □

**Lemma 9.** *Let $m \le \omega$ and $\kappa$ be a cardinal with uncountable cofinality. Let $X$ be a pointed metric space with a pseudo-translation and assume that*

*$X$ is locally path-connected at the origin $\theta$ of $X$. The space $X$ has the $\kappa$-discrete $m$-cells property if and only if each neighborhood $U$ of $\theta$ has density* $\mathrm{dens}(U) \geq \kappa$.

*Proof.* This lemma will follow from Lemmas 7 and 8 as soon as we check that each neighborhood $U \subset X$ of the origin $\theta$ contains a separated subset $S$ of size $|S| \geq \kappa$. Assuming that this is not true, for every $n \in \omega$ we may select a maximal $2^{-n}$-separated subset $S_n \subset U$ and conclude that $|S_n| < \kappa$. Then the union $S = \bigcup_{n\in\omega} S_n$ is a dense subset in $U$ and has size $|S| \leq \sum_{n\in\omega} |S_n| < \kappa$ because $\kappa$ has uncountable cofinality. This implies that $\mathrm{dens}(U) < \kappa$, which contradicts our hypothesis. □

Lemma 3 implies that the preceding lemma is true for any cardinal $\kappa$ if the space $X$ has $\omega$-LFAP.

**Proposition 1.** *Let $m, \kappa$ be cardinals. A pointed metric space $X$ with a pseudo-translation has the $\kappa$-discrete $m$-cells property provided $X$ is locally path-connected at the origin $\theta$ of $X$, $X$ has $\omega$-LFAP, and each neighborhood $U \subset X$ of the origin $\theta$ has density* $\mathrm{dens}(U) \geq \kappa$.

## 7. Recognizing the Hilbert space topology in spaces with an algebraic structure

In this final section we shall apply the obtained result to recognize the Hilbert space topology of some spaces endowed with a compatible algebraic structure.

First note the following corollary to Proposition 1 and Theorem 7.

**Corollary 1.** *A pointed metric space $X$ with a pseudo-translation is a Hilbert manifold if $X$ is a completely metrizable absolute neighborhood retract with $\omega$-LFAP and each neighborhood $U \subset X$ of the origin $\theta$ of $X$ has density equal to the density of $X$.*

Since each first countable topological group $G$ admits a left-invariant metric and each submonoid of $G$ endowed with this metric has a pseudo-translation, we get

**Corollary 2.** *A submonoid $X$ of a first countable topological group $G$ is a Hilbert manifold if $X$ is a completely metrizable absolute neighborhood retract with $\omega$-LFAP and each neighborhood $U \subset X$ of the neutral element $\theta$ of $X$ has density equal to the density of $X$.*

The following corollary to Corollary 2 yields a part of Theorem 5.

**Corollary 3.** *A topological group $X$ is an infinite-dimensional Hilbert manifold if and only if $X$ is a completely metrizable absolute neighborhood retract with $\omega$-LFAP.*

*Proof.* The "only if" part follows from the Toruńczyk's characterization Theorem 7. To prove the "if" part, assume that $G$ is a completely metrizable absolute neighborhood retract with $\omega$-LFAP. Then $G$ is locally path-connected and hence the connected component $G_0$ of the neutral element $\theta$ is an open-and-closed subgroup of $G$. Since $G$ is a discrete union of translation copies of $G_0$, it suffices to prove that $G_0$ is an infinite-dimensional Hilbert manifold. This will follow from Corollary 2 as soon as we check that each neighborhood $U \subset G_0$ of $\theta$ has density equal to the density of $G_0$. Given a neighborhood $U \subset G_0$ of $\theta$, consider the open cover $\mathcal{U} = \{gU : g \in G_0\}$ of $G_0$ by translations of $U$. By the paracompactness of $G_0$ there is a locally finite open cover $\mathcal{V}$ of $G_0$ inscribed into $\mathcal{U}$. It is clear that each element of $\mathcal{V}$ has density $\le \mathrm{dens}(U)$. Now consider the sets $X_0 = \{\theta\}$ and $X_{n+1} = \cup \mathcal{S}t(X_n, \mathcal{V})$ for $n \ge 0$. By induction we can show that each set $X_n$ has density $\le \mathrm{dens}(U)$ and so does the union $X_\omega = \bigcup_{n\in\omega} X_n$. The set $X_\omega$, being closed-and-open in the connected space $G_0$, coincides with $G_0$. Consequently, $\mathrm{dens}(G_0) = \mathrm{dens}(X_\omega) = \mathrm{dens}(U)$. □

Next we study the topology of convex sets in linear metric spaces.

**Proposition 2.** *A convex subset $X$ of a linear metric space $L$ has the $\kappa$-discrete $m$-cells property for every cardinal $m$ and every cardinal $\kappa \le$ $\mathrm{dens}(X)$ of uncountable cofinality.*

*Proof.* Given a convex set $X \subset L$ consider the convex cone

$$S = \{(tx, t) : x \in X,\ t \in [0, +\infty)\} \subset L \times \mathbb{R}$$

in $L \times \mathbb{R}$ with base $X \times \{1\}$ which will be identified with $X$. The cone $S$ is a submonoid of $L \times \mathbb{R}$ and $\theta = (0, 0)$ is the neutral element of $S$.

By $\mathrm{pr} : S \to \mathbb{R}_+$, $\mathrm{pr} : (x, t) \mapsto t$, we denote the projection onto the second coordinate. Observe that the map $r : S \setminus \{\theta\} \to X$, $r : (x, t) \mapsto x/t$, determines a retraction of $S \setminus \{\theta\}$ onto $X$. This retraction restricted to the set $S_{[\frac13,3]} = \mathrm{pr}^{-1}([\frac13, 3])$ is a perfect map.

To prove that $X$ has the $\kappa$-discrete $m$-cells property for a cardinal $\kappa \le \mathrm{dens}(X)$ of uncountable cofinality, fix an open cover $\mathcal{U}$ of $X$ and a map $f : \kappa \times I^m \to X$. According to Lemma 2 it suffices to find a map $\tilde f : \kappa \times I^m \to X$ which is $\mathcal{U}$-near to $f$ and such that the family $\{\tilde f(\{\alpha\} \times I^m)\}_{\alpha<\kappa}$ is locally finite in $X$. For each open set $U \in \mathcal{U}$ consider the set $\tilde U = \{(tx, t) : x \in U,\ \frac13 < t < 3\}$. Then $\tilde{\mathcal{U}} = \{\mathrm{pr}^{-1}(\mathbb{R} \setminus [\frac12, 2]), \tilde U : U \in \mathcal{U}\}$ is an open cover of $S$.

It follows from $\kappa \le \mathrm{dens}(X)$ that $\mathrm{dens}(X)$ is uncountable and so is $\mathrm{dens}(S) = \mathrm{dens}(X)$. Moreover, the convexity of $S$ implies that $\mathrm{dens}(W) = \mathrm{dens}(S)$ for every neighborhood $W \subset S$ of $\theta$. Applying Lemma 9 to the space $S$, we conclude that $S$ has the $\kappa$-discrete $m$-cells property, which allows us to find a map $g : \kappa \times I^m \to S$ such that $g$ is $\tilde{\mathcal{U}}$-near to $f$ and $(g(\{\alpha\} \times I^m))_{\alpha\in\kappa}$ is discrete in $S$. It follows that $g(\kappa \times I^m) \subset S \setminus \{(0, 0)\}$, so we can consider the composition $\tilde f = r \circ g$ which is $\mathcal{U}$-near to $f$ by the choice of the cover $\tilde{\mathcal{U}}$. Moreover, it follows from $g(\kappa \times I^m) \subset S_{[\frac13,3]}$ and the perfectness of $r|S_{[\frac13,3]}$ that the family $\{\tilde f(\{\alpha\} \times I^m)\}_{\alpha<\kappa}$ is locally finite in $X$. □

Combining this proposition with Lemma 3 and Theorem 7, we get the second part of Theorem 5.

**Corollary 4.** *A convex subset $X$ of a linear metric space is homeomorphic to an infinite-dimensional Hilbert space if and only if $X$ is a completely metrizable absolute retract with $\omega$-LFAP.*

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