

# CHARACTERIZING METRIC SPACES WHOSE HYPERSPACES ARE HOMEOMORPHIC TO $\ell_2$

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ABSTRACT. It is shown that the hyperspace of nonempty (bounded) closed subsets  $\text{Cld}_H(X)$  ( $\text{Bdd}_H(X)$ ) of a metric space  $(X, d)$  is homeomorphic to  $\ell_2$  if and only if the completion  $\overline{X}$  of  $X$  is connected, and locally connected,  $X$  is topologically complete and nowhere locally compact, and each (bounded) subset of  $X$  is totally bounded.

## 1. INTRODUCTION

In this paper we characterize metric spaces  $X$  whose hyperspaces  $\text{Cld}_H(X)$  and  $\text{Bdd}_H(X)$  of closed and closed bounded subsets are homeomorphic to the separable Hilbert space  $\ell_2$ . For a metric space  $(X, d)$  by  $\text{Cld}_H(X)$  we denote the space of non-empty closed subsets of  $X$  endowed with the topology generated by the Hausdorff “metric”

$$d_H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}.$$

For an unbounded metric space  $(X, d)$  the “metric”  $d_H$  can accept the infinite value but still generates a topology on  $\text{Cld}_H(X)$  called the Hausdorff topology. More precisely, this topology is generated by the metric  $\min\{1, d_H\}$ . By  $\text{Bdd}_H(X)$  we denote the subspace of  $\text{Cld}_H(X)$  consisting of closed bounded subsets of the metric space  $(X, d)$ . The hyperspace  $\text{Cld}_H(X)$  is a classical object of topology and has applications in Set-Valued Analysis, see e.g., [1]. For a compact metric space  $X$  the Hausdorff topology on  $\text{Cld}_H(X)$  coincides with the Vietoris topology, another classical topology on  $\text{Cld}(X)$ , see [9, 2.7.20] (cf. [2]). More generally, the Vietoris topology coincides with the Hausdorff topology on the subspace  $\text{Comp}(X)$  of  $\text{Cld}(X)$  consisting of non-empty compact subsets of  $X$ , see [9, 8.5.16(c)].

One of the brightest results concerning the topology of hyperspaces is the famous Curtis-Shori Theorem [5] characterizing non-degenerate Peano continua as metric spaces  $X$  whose hyperspace  $\text{Cld}_H(X)$  is homeomorphic to the Hilbert cube  $Q = [0, 1]^\omega$ . The next step in this direction was made by D. Curtis who proved in [7] that the hyperspace  $\text{Comp}(X)$  of non-empty compact subsets of a metric space  $X$  is homeomorphic to  $Q \times [0, 1]$  if and only if  $X$  is non-compact, locally compact, connected, and locally connected. Another result of D. Curtis [6] states that the hyperspace  $\text{Comp}(X)$  is homeomorphic to the separable Hilbert space  $\ell_2$  if and only if  $X$  is connected, locally connected, topologically complete and nowhere locally compact. We recall that a space  $X$  is *topologically complete* if  $X$  is homeomorphic to a complete metric space.

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In this paper we characterize metric spaces  $X$  whose hyperspaces  $\text{Cld}_H(X)$  and  $\text{Bdd}_H(X)$  are homeomorphic to  $\ell_2$ . We call a metric space  $X$  *proper* if each closed bounded subset of  $X$  is compact.

**Theorem 1.** *The hyperspace  $\text{Bdd}_H(X)$  (resp.  $\text{Cld}_H(X)$ ) of a metric space  $(X, d)$  is homeomorphic to  $\ell_2$  if and only if  $X$  is a topologically complete nowhere locally compact space and the completion  $\overline{X}$  of  $X$  is proper (resp. compact), connected, and locally connected.*

Applying this theorem to the metric spaces  $\mathbb{R} \setminus \mathbb{Q}$  and  $I \setminus \mathbb{Q}$  of irrational numbers on the real line and the interval  $I = [0, 1]$ , we obtain the following result, second part of which is due to W.Kubiś and K.Sakai [10].

**Corollary 1.** *The hyperspaces  $\text{Cld}_H(I \setminus \mathbb{Q})$  and  $\text{Bdd}_H(\mathbb{R} \setminus \mathbb{Q})$  are homeomorphic to  $\ell_2$ .*

Let us remark that unlike the hyperspaces  $\text{Cld}_H(I \setminus \mathbb{Q})$  and  $\text{Bdd}_H(\mathbb{R} \setminus \mathbb{Q})$  the hyperspace  $\text{Cld}_H(\mathbb{R} \setminus \mathbb{Q})$  is not homeomorphic to  $\ell_2$  since it is neither connected nor separable.

As a by-product of the proof of Theorem 1 we obtain the following characterizations of metric spaces whose hyperspaces are separable absolute retracts.

**Theorem 2.** *The hyperspace  $\text{Bdd}_H(X)$  (resp.  $\text{Cld}_H(X)$ ) of a metric space  $X$  is a separable AR if and only if the completion  $\overline{X}$  of  $X$  is proper (resp. compact), connected and locally connected.*

## 2. HOMOTOPY DENSE SUBSETS IN THE HILBERT CUBE

A subset  $Y$  of a topological space  $X$  is *homotopy dense* in  $X$  if there is a homotopy  $(h_t)_{t \in I} : X \rightarrow X$  such that  $h_0 = \text{id}$  and  $h_t(X) \subset Y$  for every  $t > 0$ . The following lemma detecting topological copies of  $\ell_2$  in the Hilbert cube  $Q$  is due to D. Curtis [8] and is our main tool in the proof of Theorem 1.

**Lemma 1.** *A homotopy dense  $G_\delta$ -subset  $X \subset Q$  with homotopy dense complement in the Hilbert cube  $Q$  is homeomorphic to  $\ell_2$ .*

## 3. TOPOLOGY OF LAWSON SEMILATTICES

Theorem 2 will be derived from a more general result concerning Lawson semilattices. By a *topological semilattice* we understand a pair  $(L, \vee)$  consisting of a topological space  $L$  and a continuous associative commutative idempotent operation  $\vee : L \times L \rightarrow L$ . A topological semilattice  $(L, \vee)$  is a *Lawson semilattice* if open subsemilattices form a base of the topology of  $L$ . A typical example of a Lawson semilattice is the hyperspace  $\text{Cld}_H(X)$  endowed with the operation of union  $\cup$ , see [13, 5.4].

Each semilattice  $(L, \vee)$  carries a natural partial order:  $x \leq y$  iff  $x \vee y = y$ . A semilattice  $(L, \vee)$  is called *complete* if each subset  $A \subset L$  has the smallest upper bound  $\sup A \in L$ . It is well-known (and can be easily proved) that each compact topological semilattice is complete.

**Lemma 2.** *If  $L$  is a locally compact Lawson semilattice, then each compact subset  $K \subset L$  has the smallest upper bound  $\sup K \in L$ . Moreover, the map  $\text{sup} : \text{Comp}(L) \rightarrow L$ ,  $\text{sup} : K \mapsto \sup K$ , is a continuous semilattice homomorphism. Also for every subset  $A \subset L$  with compact closure  $\overline{A}$  we have  $\sup A = \sup \overline{A}$ .*

This lemma easily follows from its compact version proved by J. Lawson in [12].

In Lawson semilattices many geometric questions reduce to the one-dimensional level. The following fact illustrating this phenomenon is proved in [13].

**Lemma 3.** *Let  $X$  be a dense subsemilattice of a metrizable Lawson semilattice  $L$ . If  $X$  is relatively  $LC^0$  in  $L$  (and  $X$  is path-connected), then  $X$  and  $L$  are ANRs (ARs) and  $X$  is homotopy dense in  $L$ .*

A subset  $Y \subset X$  is defined to be *relatively  $LC^0$*  in  $X$  if for every  $x \in X$ , each neighborhood  $U$  of  $x$  in  $X$  contains a smaller neighborhood  $V$  of  $x$  such that every two points of  $V \cap Y$  can be joined by a path in  $U \cap Y$ .

Under a suitable completeness condition, the density of a subsemilattice is equivalent to the homotopical density.

A subsemilattice  $X$  of semilattice  $L$  is defined to be *relatively complete* in  $L$  if for any subset  $A \subset X$  having the smallest upper bound  $\sup A$  in  $L$  this bound belongs to  $X$ .

**Proposition 1.** *Let  $L$  be a metrizable locally compact locally connected Lawson semilattice. Each dense relatively complete subsemilattice  $X \subset L$  is homotopy dense in  $L$ .*

*Proof.* According to Lemma 3 it suffices to check that  $X$  is relatively  $LC^0$  in  $L$ . Given a point  $x_0 \in L$  and a neighborhood  $U \subset L$  of  $x_0$ , consider the canonical retraction  $\text{sup} : \text{Comp}(L) \rightarrow L$ . The space  $L$ , being locally compact and locally connected, is locally path-connected, see [11, §50.II]. By Lemma 3, the Lawson semilattice  $L$  is an ANR. Using the continuity of  $\text{sup}$ , find a path-connected neighborhood  $V \subset L$  of  $x_0$  such that  $\text{sup}(\text{Comp}(\overline{V})) \subset U$ . We claim that any two points  $x, y \in X \cap V$  can be connected by a path in  $X \cap U$ . First we construct a path  $\gamma : [0, 1] \rightarrow \overline{V}$  such that  $\gamma(0) = x$ ,  $\gamma(1) = y$  and  $\gamma^{-1}(X)$  is dense in  $[0, 1]$ . Let  $\{q_n : n \in \omega\}$  be a countable dense subset in  $[0, 1]$  with  $q_0 = 0$  and  $q_1 = 1$ . The space  $L$ , being locally compact, admits a complete metric  $\rho$ . The path-connectedness of  $V$  implies the existence of a continuous map  $\gamma_0 : [0, 1] \rightarrow V$  such that  $\gamma_0(0) = x$  and  $\gamma_0(1) = y$ . Using the local path-connectedness of  $L$  we can construct inductively a sequence of functions  $\gamma_n : [0, 1] \rightarrow V$  such that

- $\gamma_n(q_k) = \gamma_{n-1}(q_k)$  for all  $k \leq n$ ;
- $\gamma_n(q_{n+1}) \in X$ ;
- $\sup_{t \in [0, 1]} \rho(\gamma_n(t), \gamma_{n-1}(t)) < 2^{-n}$ .

Then the map  $\gamma = \lim_{n \rightarrow \infty} \gamma_n : [0, 1] \rightarrow \overline{V}$  is continuous and has the desired properties:  $\gamma(0) = x$ ,  $\gamma(1) = y$  and  $\gamma(q_n) \in X$  for all  $n \in \omega$ .

For every  $t \in [0, 1]$  consider the set  $\Gamma(t) = \{\gamma(s) : |t - s| \leq \text{dist}(t, \{0, 1\})\}$ . It is clear that the map  $\Gamma : [0, 1] \rightarrow \text{Comp}(L)$  is continuous and so is the composition  $\text{sup} \circ \Gamma : [0, 1] \rightarrow L$ . Observe that  $\text{sup} \circ \Gamma(0) = \text{sup}\{\gamma(0)\} = \gamma(0) = x$ ,  $\text{sup} \circ \Gamma(1) = y$ , and  $\text{sup} \circ \Gamma([0, 1]) \subset \text{sup}(\text{Comp}(\overline{V})) \subset U$ . Since for every  $t \in (0, 1)$  the set  $\Gamma(t) = \overline{\Gamma(t) \cap X}$ , we get  $\text{sup} \Gamma(t) = \text{sup}(\Gamma(t) \cap X) \in X$  by the relative completeness of  $X$  in  $L$ . Thus  $\text{sup} \circ \Gamma : [0, 1] \rightarrow U \cap X$  is a path connecting  $x$  and  $y$  in  $U$ .  $\square$

#### 4. SOME TOPOLOGICAL PROPERTIES OF HYPERSPACES

In this section we collect some easy (and known) lemmas that will be used in the subsequent proofs.

**Lemma 4.** *For a metric space  $X$  the following conditions are equivalent:*

- (1)  $X$  is topologically complete;
- (2)  $\text{Cld}_H(X)$  is topologically complete;
- (3)  $\text{Bdd}_H(X)$  is topologically complete.

**Lemma 5.** *For a metric space  $X$  the following conditions are equivalent:*

- (1)  $X$  is nowhere locally compact;
- (2)  $\text{Cld}_H(X)$  is nowhere locally compact;
- (3)  $\text{Bdd}_H(X)$  is nowhere locally compact.

**Lemma 6.** *Let  $X$  be a metric space. The hyperspace  $\text{Cld}_H(X)$  (resp.  $\text{Bdd}_H(X)$ ) is separable if and only if each (bounded) subset of  $X$  is totally bounded.*

The following lemma is not trivial and can be found in [3, 3.7].

**Lemma 7.** *Let  $X$  be a dense subspace of a metric space  $M$ . The hyperspace  $\text{Cld}_H(X)$  (resp.  $\text{Bdd}_H(X)$ ) is an absolute retract if and only if so is the hyperspace  $\text{Cld}_H(M)$  (resp.  $\text{Bdd}_H(M)$ ).*

For a metric space  $X$  by  $\text{Fin}(X)$  we denote the subspace of  $\text{Comp}(X)$  consisting of non-empty finite subspaces of  $X$ .

**Lemma 8.** *If  $Y$  is a subset of a locally path-connected space  $X$ , then the subset  $L = \text{Fin}(X) \setminus \text{Fin}(Y)$  is relatively  $LC^0$  in  $\text{Comp}(X)$ .*

*Proof.* By the argument of [4] we can show that  $\text{Fin}(X)$  is relatively  $LC^0$  in  $\text{Comp}(X)$ . Consequently, for every compact set  $K \in \text{Comp}(X)$  and a neighborhood  $U \subset \text{Comp}(X)$  of  $K$  there is a neighborhood  $V \subset \text{Comp}(X)$  of  $K$  such that any two points  $A, B \in \text{Fin}(X) \cap V$  can be linked by a path in  $\text{Fin}(X) \cap U$ . Since  $\text{Comp}(X)$  is a Lawson semilattice, we may assume that  $U$  and  $V$  are subsemilattices of  $\text{Comp}(X)$ . We claim that any two points  $A, B \in L \cap V$  can be connected by a path in  $L \cap U$ . Since  $L \subset \text{Fin}(X)$ , there is a path  $\gamma : [0, 1] \rightarrow U \cap \text{Fin}(X)$  such that  $\gamma(0) = A$  and  $\gamma(1) = B$ . Define a new path  $\gamma' : [0, 1] \rightarrow U \cap \text{Fin}(X)$  letting  $\gamma'(t) = \gamma(\max\{0, 2t - 1\}) \cup \gamma(\min\{2t, 1\})$ . Observe that  $A \subset \gamma'(t)$  if  $t \leq 1/2$  and  $B \subset \gamma'(t)$  if  $t \geq 1/2$ . Since  $A, B \notin \text{Fin}(Y)$ , we conclude that  $\gamma'([0, 1]) \subset L \cap U$ .  $\square$

## 5. PROOF OF THEOREM 2

Let  $X$  be a metric space and  $\overline{X}$  be its completion. First we prove that  $\text{Bdd}_H(X)$  is a separable AR if and only if  $\overline{X}$  is proper, connected and locally connected.

To prove the “only if” part, assume that  $\text{Bdd}_H(X)$  is a separable absolute retract. By Lemma 7, the hyperspace  $\text{Bdd}_H(\overline{X})$  is a separable absolute retract too. By Lemma 6, the separability of  $\text{Bdd}_H(X)$  implies that each bounded subset of  $X$  is totally bounded, which is equivalent to the properness of the completion  $\overline{X}$  of  $X$ . In this case  $\text{Comp}(\overline{X}) = \text{Bdd}_H(\overline{X})$  is an absolute retract and we can apply the Curtis Theorem [7] to conclude that the locally compact space  $\overline{X}$  is connected and locally connected.

Next, we prove the “if” part of Theorem 2. Assume that the completion  $\overline{X}$  of  $X$  is proper, connected, and locally connected. Then  $\text{Bdd}_H(\overline{X}) = \text{Comp}(\overline{X})$  is a separable locally compact absolute retract by [7]. The subsemilattice  $\text{Bdd}_H(X)$ , being relatively complete in  $\text{Bdd}_H(\overline{X})$  is homotopy dense in  $\text{Bdd}_H(\overline{X})$  by Proposition 1.

Now we prove that  $\text{Cld}_H(X)$  is a separable AR if and only if  $\overline{X}$  is compact, connected and locally connected.

If  $\overline{X}$  is compact, connected, and locally connected, then  $\text{Cld}_H(X) = \text{Bdd}_H(X)$  is a separable AR by the preceding case. Conversely, if  $\text{Cld}_H(X)$  is separable AR, then Lemma 6 guarantees that  $X$  is totally bounded, and hence  $\text{Cld}_H(X) = \text{Bdd}_H(X)$  and we can apply the preceding case to conclude that  $\overline{X}$  is connected and locally connected. Also the space  $\overline{X}$  is compact, being the completion of a totally bounded metric space  $X$ .

## 6. PROOF OF THEOREMS 1

Let  $X$  be a metric space. If  $\text{Bdd}_H(X)$  (resp.  $\text{Cld}_H(X)$ ) is homeomorphic to  $\ell_2$ , then  $X$  is topologically complete and nowhere locally compact by Lemmas 4 and 5. Since  $\ell_2$  is a separable AR, we may apply Theorem 2 to conclude that the completion  $\overline{X}$  of  $X$  is connected, locally connected, and proper (resp. compact). This proves the “only if” part of Theorem 1.

To prove the “if” part, assume that  $X$  is topologically complete and nowhere locally compact and the completion  $\overline{X}$  of  $X$  is proper, connected and locally connected. First we consider the case of compact  $\overline{X}$ . By Curtis-Shori Theorem [5], the hyperspace  $\text{Cld}_H(\overline{X}) = \text{Comp}(\overline{X})$  is homeomorphic to  $Q$ . Now consider the map  $e : \text{Cld}_H(X) \rightarrow \text{Cld}_H(\overline{X})$  assigning to each closed subset  $F \subset X$  its closure  $\overline{F}$  in  $\overline{X}$  and note that this map is an isometric embedding, which allows us to identify the hyperspace  $\text{Cld}_H(X)$  with the subspace  $\{F \in \text{Cld}_H(\overline{X}) : F = \text{cl}(F \cap X)\}$  of  $\text{Cld}_H(\overline{X})$ . It is easy to check that this subspace is dense and relatively complete in the Lawson semilattice  $\text{Cld}_H(\overline{X})$ . Then it is homotopically dense in  $\text{Cld}_H(\overline{X})$  by Proposition 1 and Lemma 3. By Lemma 4, the subset  $\text{Cld}_H(X)$ , being topologically complete, is a  $G_\delta$ -set in  $\text{Cld}_H(\overline{X})$ . Since  $X$  is nowhere locally compact, the complement  $\overline{X} \setminus X$  is dense in  $\overline{X}$ . By Lemmas 4 and 8, the dense subsemilattice  $L = \text{Fin}(\overline{X}) \setminus \text{Fin}(X)$  is homotopy dense in  $\text{Cld}_H(\overline{X})$ . Since  $L \cap \text{Cld}_H(X) = \emptyset$ , we get that  $\text{Cld}_H(X)$  is a homotopy dense  $G_\delta$ -subset in  $\text{Cld}_H(\overline{X})$  with homotopy dense complement. Applying Lemma 1 we conclude that the space  $\text{Cld}_H(X)$  is homeomorphic to  $\ell_2$ .

Next, we consider the case of non-compact  $\overline{X}$ . It follows from the properness of  $\overline{X}$  that  $\text{Bdd}_H(\overline{X}) = \text{Comp}_H(\overline{X})$  and hence  $\text{Bdd}_H(\overline{X})$  is homeomorphic to the Hilbert cube with deleted point  $Q \setminus \{pt\}$  by the Curtis Theorem [7]. Repeating the preceding argument, we can prove that  $\text{Bdd}_H(X)$  can be identified with homotopy dense  $G_\delta$ -set with homotopy negligible complement in  $\text{Bdd}_H(\overline{X})$ . Since the one-point compactification of  $\text{Bdd}_H(\overline{X})$  is homeomorphic to the Hilbert cube, we can apply Lemma 1 to conclude that  $\text{Bdd}_H(X)$  is homeomorphic to  $\ell_2$ .

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