

Submitted to
Topology Proceedings

**CLOSED LOCALLY PATH-CONNECTED
SUBSPACES OF FINITE-DIMENSIONAL GROUPS
ARE LOCALLY COMPACT**

TARAS BANAKH AND LYUBOMYR ZDOMSKYY

ABSTRACT. We prove that each closed locally continuum-connected subspace of a finite dimensional topological group is locally compact. This allows us to construct many 1-dimensional metrizable separable spaces that are not homeomorphic to closed subsets of finite-dimensional topological groups, which answers in negative a question of D.Shakhmatov. Another corollary is a characterization of Lie groups as finite-dimensional locally continuum-connected topological groups. For locally path connected topological groups this characterization was proved by Gleason and Palais in 1957.

1. INTRODUCTION

It follows from the classical Menger-Nöbeling-Pontryagin Theorem that each separable metrizable space X of dimension $n = \dim X < \infty$ admits a topological embedding $e : X \rightarrow G$ into a metrizable separable group G of dimension $\dim(G) = 2n + 1$ (for such a group G we can take the $(2n + 1)$ -dimensional Euclidean space \mathbb{R}^{2n+1}). In [14] (see also [4, Question 7]) Dmitri Shakhmatov asked if we can additionally require of $e : X \rightarrow G$ to be a **closed** embedding? In this paper we shall give a strongly negative answer to this question constructing simple 1-dimensional spaces that are

2000 *Mathematics Subject Classification.* 54H11, 54F45, 54F15.

Key words and phrases. Locally continuum-connected, finite-dimensional, (closed) embedding, topological group.

not homeomorphic to closed subspaces of finite-dimensional topological groups.

First we state a Key Lemma treating locally continuum-connected subspaces of finite-dimensional topological groups. By a *continuum* we understand a connected compact Hausdorff space. We shall say that two points x, y of a topological space X are connected by a *subcontinuum* $K \subset X$ if $x, y \in K$.

Following [3] we define a topological space X to be *locally continuum-connected* at a point $x \in X$ if for every neighborhood $U \subset X$ of x there is another neighborhood $V \subset U$ of x such that each point $y \in V$ can be connected with x by a subcontinuum $K \subset U$. A space X is *locally continuum-connected* if it is locally continuum-connected at each point. It is clear that X is locally continuum-connected at $x \in X$ if X is locally path-connected at x .

The “locally path-connected” version of the following lemma was proved by A.Gleason [6] and D.Montgomery [12] and then was used by A.Gleason and R.Palais [7] to prove that locally path-connected finite-dimensional topological groups are Lie groups. We shall say that a topological group G is *compactly finite-dimensional* if

$$\text{co-dim}(G) = \sup\{\dim(K) : K \text{ is a compact subspace of } G\}$$

is finite. Observe that a subgroup of a compactly finite-dimensional group also is compactly finite-dimensional.

Lemma 1.1 (Key Lemma). *If a subspace X of a compactly finite-dimensional topological group G is locally continuum-connected at a point $x \in X$, then some neighborhood $U \subset X$ of x has compact closure in G .*

Because of its technical character, we postpone the proof of this lemma till the end of the paper. Now we state several corollaries of this lemma. We recall that a regular topological space X is *cosmic* if it is a continuous image of a separable metrizable space. This is equivalent to the countability of the network weight of X .

Theorem 1.2. *Each (closed) locally continuum-connected cosmic subspace X of a compactly finite-dimensional topological group G is metrizable (and locally compact).*

Proof. Replacing G by the group hull of X , if necessary, we may assume that X algebraically generates G . Then G is a cosmic

space and hence all compact subsets of G are metrizable. By Key Lemma, each point $x \in X$ has a neighborhood $U \subset X$ with compact (and thus metrizable) closure \overline{U} in G . If X is closed in G , then $U = \overline{U}$ is compact and hence X is locally compact. Being locally metrizable and paracompact (because of Lindelöf), the space X is metrizable. \square

Corollary 1.3. *For a finite-dimensional locally continuum-connected cosmic space X the following conditions are equivalent:*

- (1) X admits a closed embedding into a compactly finite-dimensional topological group;
- (2) X admits a closed embedding into \mathbb{R}^{2n+1} where $n = \dim(X)$;
- (3) X is metrizable and locally compact.

This corollary supplies us with many examples of finite-dimensional second countable spaces admitting no closed embedding into (compactly) finite-dimensional topological groups.

Probably the simplest one is the hedgehog

$$H_\omega = \bigcup_{n \in \omega} [0, 1] \cdot \vec{e}_n \subset l_2$$

where (\vec{e}_n) is the standard orthonormal basis in the Hilbert space l_2 .

Corollary 1.4. *No compactly finite-dimensional topological group contains a closed subspace homeomorphic to the hedgehog H_ω .*

Besides the metrizable topology the hedgehog carries also a natural non-metrizable topology, namely the strongest topology inducing the original Euclidean topology on each needle $[0, 1] \cdot \vec{e}_n$, $n \in \omega$. The hedgehog endowed with this non-metrizable topology will be denoted by V_ω and will be referred to as the *Fréchet-Urysohn hedgehog* (by analogy with the Fréchet-Urysohn fan).

The metrizable hedgehog H_ω admits no closed embedding into a compactly finite-dimensional group, but can be embedded into the 2-dimensional group \mathbb{R}^2 . In contrast with this, by Theorem 1.2, nothing similar cannot be done for the Fréchet-Urysohn hedgehog V_ω .

Corollary 1.5. *No compactly finite-dimensional topological group contains a subspace homeomorphic to the Fréchet-Urysohn hedgehog V_ω .*

Remark 1.6. By Corollaries 1.4 and 1.5, a topological group G containing a topological copy of V_ω **or** a closed topological copies of H_ω is not finite-dimensional. By its form, this results resembles a result of [2] or [1]: a topological group with countable pseudocharacter containing a topological copy of V_ω **and** a closed topological copy of H_ω is not sequential. We do not know if this resemblance is occasional.

It is also interesting to compare Corollaries 1.4 and 1.5 with a result of J.Kulesza [Kul] who proved that the metric hedgehog H_{ω_1} with ω_1 spines cannot be embedded into a finite-dimensional topological group. The other his result says that the hedgehog H_3 with three spines cannot be embedded into a 1-dimensional topological group.

Key Lemma has another interesting corollary related to the famous fifth problem of Hilbert.

Corollary 1.7. *A topological group G is a Lie group if and only if G is compactly finite-dimensional and locally continuum-connected.*

Proof. The “only if” part is trivial. To prove the “if” part, take any locally continuum-connected compactly finite-dimensional topological group G . Key Lemma implies that G is locally compact and hence finite-dimensional. By [13, p.185], the group G is a Lie group, being locally compact, locally connected, and finite-dimensional. \square

This corollary generalizes Gleason-Palais Theorem [7] stating that each locally path-connected finite-dimensional topological group is a Lie group. The Gleason-Palais Theorem was applied by S. N. Hudson [9] to show that each path-connected locally connected finite-dimensional topological group is a Lie group.

Problem 1.8. *Let G be a locally connected finite-dimensional topological group. Is G a Lie group if G is continuum-connected? (The latter means that any two points of G can be connected by a subcontinuum of G).*

It should be mentioned that there exists a connected locally connected subgroup of \mathbb{R}^2 that contains no arc, see [10].

2. A DIMENSION LEMMA

The proof of Key Lemma relies on the following (probably known) fact from Dimension Theory.

Lemma 2.1. *Let K_1, \dots, K_n be continua and for every $i \leq n$ let $a_i, b_i \in K_i$ be two distinct points. Let $K = \prod_{i=1}^n K_i$ and $A_i = \text{pr}_i^{-1}(a_i)$, $B_i = \text{pr}_i^{-1}(b_i)$ where $\text{pr}_i : K \rightarrow K_i$ is the projection. Let $f : K \rightarrow X$ be a continuous map to a Hausdorff topological space such that $f(A_i) \cap f(B_i) = \emptyset$ for all $i \leq n$. Then $\dim f(K) \geq n$.*

Proof. Assume conversely that $\dim f(K) < n$ and apply Theorem on Partitions [5, 3.2.6] to find closed subsets P_1, \dots, P_n of $f(K)$ such that $\bigcap_{i=1}^n P_i = \emptyset$ and each P_i is a partition between $f(A_i)$ and $f(B_i)$ in $f(K)$ (the latter means that A_i and B_i lie in different connected components of $f(K) \setminus P_i$).

Then for every $i \leq n$ the set $L_i = f^{-1}(P_i)$ is a partition between A_i and B_i in K and $\bigcap_{i=1}^n L_i = \emptyset$. This means that the sequence $(A_1, B_1), \dots, (A_n, B_n)$ is inessential in $K = \prod_{i=1}^n K_i$, which contradicts the results of Holsztyński [8] or Lifanov [11], see also [5, 1.8.K]. \square

3. PROOF OF KEY LEMMA

Let G be a topological group with $n = \text{co-dim}(G) < \infty$ and X be a subspace of G that is locally continuum-connected at a point $e \in X$. We have to show that e has a neighborhood $U \subset X$ with compact closure in G .

Assuming the converse we shall derive a contradiction. Without loss of generality, the point e is the neutral element of the group G . By finite induction, we shall construct a sequence $(C_i)_{i=0}^n$ of subcontinua of $X \subset G$ connecting e with some other point $a_i \in X \setminus \{e\}$ such that for the ‘‘cube’’ $K_i = \prod_{j \leq i} C_j$ and the projections $\text{pr}_{k,i} : \prod_{j \leq i} C_j \rightarrow C_k$, $k \leq i$, the ‘‘faces’’ $A_{k,i} = \text{pr}_{k,i}^{-1}(e)$, $B_{k,i} = \text{pr}_{k,i}^{-1}(a_k)$ have disjoint images under the map

$$f_i : \prod_{j \leq i} C_j \rightarrow G, \quad f_i : (x_0, \dots, x_i) \mapsto x_0 \cdots x_i.$$

By Dimension Lemma 2.1 this will imply

$$n = \text{co-dim } G \geq \dim f_i(K_i) \geq i + 1,$$

which is not possible for $i = n$.

Assume that for some $i \leq n$ the pointed continua (C_j, a_j) , $j < i$, have been constructed, so that for every $k < i$ the “faces” $A_{k,i-1}$, $B_{k,i-1}$ have disjoint images $f_{i-1}(A_{k,i-1})$ and $f_{i-1}(B_{k,i-1})$ (we start the induction from the trivial case $i = 0$). The compactness argument yields us a neighborhood $U_i \subset G$ of e such that the sets $f_{i-1}(A_{k,i-1}) \cdot U_i$ and $f_{i-1}(B_{k,i-1}) \cdot U_i$ do not intersect for all $k < i$. By the local continuum-connectedness of X at e , there is a neighborhood $V_i \subset U_i \cap X$ of e in X such that any point $x \in V_i$ can be connected with e by a subcontinuum $C \subset U_i$. Consider the compact subset $K = \{x^{-1}y : x, y \in f_{i-1}(K_{i-1})\}$ of G . By our assumption, no neighborhood of e in X has compact closure in G . Consequently, $V_i \not\subset K$ and we can find a point $a_i \in V_i \setminus K$. By the choice of V_i , there is a compact connected subset $C_i \in U_i$ containing the points e and a_i .

To complete the inductive step it now suffices to check that

$$f_i(A_{k,i}) \cap f_i(B_{k,i}) = \emptyset$$

for all $k \leq i$.

If $k < i$, then $A_{k,i} = A_{k,i-1} \times C_i$ and $f_i(A_{k,i}) \subset f_{i-1}(A_{k,i-1}) \cdot C_i \subset f_{i-1}(A_{k,i-1}) \cdot U_i$. By analogy, $f_i(B_{k,i}) \subset f_{i-1}(B_{k,i-1}) \cdot U_i$. Consequently, the sets $f_i(A_{k,i})$ and $f_i(B_{k,i})$ do not intersect by the choice of the neighborhood U_i .

If $k = i$, then $A_{k,i} = K_{i-1} \times \{e\}$ and $B_{k,i} = K_{i-1} \times \{a_i\}$. Consequently, $f_i(A_{k,i}) = f_{i-1}(K_{i-1})$ and $f_i(B_{k,i}) = f_{i-1}(K_{i-1}) \cdot a_i$. Now the choice of the point $a_i \notin K$, implies that $f_i(A_{k,i}) \cap f_i(B_{k,i}) = \emptyset$. This finishes the inductive construction as well as the proof of Key Lemma.

4. ACKNOWLEDGEMENT

The authors would like to thank Dikran Dikranjan for valuable comments on the preliminary version of the paper.

REFERENCES

- [1] T. Banakh, *On topological groups containing a Frechet-Urysohn fan*, Mat. Stud. **9**:2 (1998), 149–154.
- [2] T. Banakh, L. Zdomskyy, *The topological structure of (homogeneous) s-spaces and groups with countable cs*-network*, Appl. Gen. Topol. **5**:1 (2004), 25–48.

- [3] D. Curtis, *Hyperspaces of non-compact metric spaces*, Compos. Math. **40** (1980), 126–130.
- [4] D. Dikranjan, D. Shakhmatov, *Selected topics from the structure theory of topological groups*, Open Problems in Topology, II (E.Pearl, ed.), Elsevier (to appear).
- [5] R. Engelking, *Theory of Dimensions, Finite and Infinite*, Heldermann Verlag, 1995.
- [6] A. Gleason, *Arcs in locally compact groups*, Proc. Natl. Acad. Sci. USA. **38** (1950), 663–667.
- [7] A. Gleason, R. Palais, *On a class of transformation groups*, Amer. J. Math. **79** (1957), 631–648.
- [8] W. Hosztyński, *Universality of mappings onto the product of snake-like spaces. Relation with dimension*. Bull. Acad. Polon. Sci. Sèr. Math. **16** (1968), 161–167.
- [9] S.N. Hudson, *On connectivity properties of finite-dimensional groups*, Proc. Amer. Math. Soc. **23** (1969), 68–72.
- [10] F. Burton Jones, *Connected and disconnected plane sets and the functional equation $f(x)+f(y) = f(x+y)$* , Bull. Amer. Math. Soc. **49** (1942), 115–120.
- [Kul] J. Kulesza, *Spaces which do not embed in topological groups of the same dimension*, Topology Appl. **50** (1993), 139–145.
- [11] I. K. Lifanov, *On the dimension of the product of bicomacta*, DAN SSSR **180** (1968), 534–537 (Engl. transl. Soviet Math. Dokl. **9** (1968), 648–651).
- [12] D. Montgomery, *Theorems on the topological structure of locally compact groups*, Ann. of Math. **50** (1949), 570–580.
- [13] D. Montgomery, L. Zippin, *Topological Transformation Groups*, Interscience New York, 1955.
- [14] D. Shakhmatov, *A survey of current reserches and open problems in the dimension theory of topological groups*, Questions Answers Gen. Topology **8:1** (1990), 101–128.

INSTYTUT MATEMATYKI, AKADEMIA ŚWIĘTOKRZYSKA, KIELCE, POLAND;
AND DEPARTMENT OF MECHANICS AND MATHEMATICS, IVAN FRANKO LVIV
NATIONAL UNIVERSITY, LVIV, UKRAINE

E-mail address: tbanakh@yahoo.com

DEPARTMENT OF MECHANICS AND MATHEMATICS, IVAN FRANKO LVIV NA-
TIONAL UNIVERSITY, UNIVERSYTETSKA1, LVIV, 79000, UKRAINE; AND DE-
PARTMENT OF MATHEMATICS, WEIZMANN INSTITUTE OF SCIENCE, REHOVOT
76100, ISRAEL

E-mail address: lzdomsky@rambler.ru