

THE TOPOLOGICAL STRUCTURE OF (HOMOGENEOUS) SPACES AND GROUPS WITH COUNTABLE cs^* -CHARACTER

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ABSTRACT. In this paper we introduce and study three new cardinal topological invariants called the cs^* -, cs -, and sb -characters. The class of topological spaces with countable cs^* -character is closed under many topological operations and contains all \aleph -spaces and all spaces with point-countable cs^* -network. Our principal result states that each non-metrizable sequential topological group with countable cs^* -character has countable pseudo-character and contains an open k_ω -subgroup. This result is specific for topological groups: under Martin Axiom there exists a sequential topologically homogeneous k_ω -space X with $\aleph_0 = cs_\chi^*(X) < \psi(X)$.

INTRODUCTION

In this paper we introduce and study three new local cardinal invariants of topological spaces called the sb -character, the cs -character and cs^* -character, and describe the structure of sequential topological groups with countable cs^* -character. All these characters are based on the notion of a *network* at a point x of a topological space X , under which we understand a collection \mathcal{N} of subsets of X such that for any neighborhood $U \subset X$ of x there is an element $N \in \mathcal{N}$ with $x \in N \subset U$, see [Lin].

A subset B of a topological space X is called a *sequential barrier* at a point $x \in X$ if for any sequence $(x_n)_{n \in \omega} \subset X$ convergent to x , there is $m \in \omega$ such that $x_n \in B$ for all $n \geq m$, see [Lin]. It is clear that each neighborhood of a point $x \in X$ is a sequential barrier for x while the converse is true for Fréchet-Urysohn spaces.

Under a *sb-network* at a point x of a topological space X we shall understand a network at x consisting of sequential barriers at x . In other words, a collection \mathcal{N} of subsets of X is a *sb-network* at x if for any neighborhood U of x there is an element N of \mathcal{N} such that for any sequence $(x_n) \subset X$ convergent to x the set N contains almost all elements of (x_n) . Changing two quantifiers in this definition by their places we get a definition of a *cs-network* at x .

Namely, we define a family \mathcal{N} of subsets of a topological space X to be a *cs-network* (resp. a *cs*-network*) at a point $x \in X$ if for any neighborhood $U \subset X$ of x and any sequence $(x_n) \subset X$ convergent to x there is an element $N \in \mathcal{N}$ such that $N \subset U$ and N contains almost all (resp. infinitely many) members of the sequence (x_n) . A family \mathcal{N} of subsets of a topological space X is called a *cs-network* (resp. *cs*-network*) if it is a *cs-network* (resp. *cs*-network*) at each point $x \in X$, see [Na].

The smallest size $|\mathcal{N}|$ of an *sb-network* (resp. *cs-network*, *cs*-network*) \mathcal{N} at a point $x \in X$ is called the *sb-character* (resp. *cs-character*, *cs*-character*) of X at the point x and is denoted by $sb_\chi(X, x)$ (resp. $cs_\chi(X, x)$, $cs_\chi^*(X, x)$). The cardinals $sb_\chi(X) = \sup_{x \in X} sb_\chi(X, x)$, $cs_\chi(X) = \sup_{x \in X} cs_\chi(X, x)$ and $cs_\chi^*(X) = \sup_{x \in X} cs_\chi^*(X, x)$ are called the *sb-character*, *cs-character* and *cs*-character* of the

topological space X , respectively. For the empty topological space $X = \emptyset$ we put $\text{sb}_\chi(X) = \text{cs}_\chi(X) = \text{cs}_\chi^*(X) = 1$.

In the sequel we shall say that a topological space X has *countable sb-character* (resp. *cs-*, *cs*-character*) if $\text{sb}_\chi(X) \leq \aleph_0$ (resp. $\text{cs}_\chi(X) \leq \aleph_0$, $\text{cs}_\chi^*(X) \leq \aleph_0$). It should be mentioned that under different names topological spaces with countable sb- or cs-character have already occurred in topological literature. In particular, a topological space has countable cs-character if and only if it is *csf*-countable in the sense of [Lin]; a (sequential) space X has countable sb-character if and only if it is universally *csf*-countable in the sense of [Lin] (if and only if it is weakly first-countable in the sense of [Ar₁] if and only if it is 0-metrizable in the sense of Nedev [Ne]). From now on, all the topological spaces considered in the paper are T_1 -spaces. At first we consider the interplay between the characters introduced above.

Proposition 1. *Let X be a topological space. Then*

- (1) $\text{cs}_\chi^*(X) \leq \text{cs}_\chi(X) \leq \text{sb}_\chi(X) \leq \chi(X)$;
- (2) $\chi(X) = \text{sb}_\chi(X)$ if X is Fréchet-Urysohn;
- (3) $\text{cs}_\chi^*(X) < \aleph_0$ iff $\text{cs}_\chi(X) < \aleph_0$ iff $\text{sb}_\chi(X) < \aleph_0$ iff $\text{cs}_\chi^*(X) = 1$ iff $\text{cs}_\chi(X) = 1$ iff $\text{sb}_\chi(X) = 1$ iff each convergent sequence in X is trivial;
- (4) $\text{sb}_\chi(X) \leq 2^{\text{cs}_\chi^*(X)}$;
- (5) $\text{cs}_\chi(X) \leq \text{cs}_\chi^*(X) \cdot \sup\{|\kappa|^{\leq \omega} : \kappa < \text{cs}_\chi^*(X)\} \leq (\text{cs}_\chi^*(X))^{\aleph_0}$ where $[\kappa]^{\leq \omega} = \{A \subset \kappa : |A| \leq \aleph_0\}$.

Here “iff” is an abbreviation for “if and only if”. The Arens’ space S_2 and the sequential fan S_ω give us simple examples distinguishing between some of the characters considered above. We recall that the *Arens’ space* S_2 is the set $\{(0, 0), (\frac{1}{n}, 0), (\frac{1}{n}, \frac{1}{nm}) : n, m \in \mathbb{N}\} \subset \mathbb{R}^2$ carrying the strongest topology inducing the original planar topology on the convergent sequences $C_0 = \{(0, 0), (\frac{1}{n}, 0) : n \in \mathbb{N}\}$ and $C_n = \{(\frac{1}{n}, 0), (\frac{1}{n}, \frac{1}{nm}) : m \in \mathbb{N}\}$, $n \in \mathbb{N}$. The quotient space $S_\omega = S_2/C_0$ obtained from the Arens’ space S_2 by identifying the points of the sequence C_0 is called the *sequential fan*, see [Lin]. The sequential fan S_ω is the simplest example of a non-metrizable Fréchet-Urysohn space while S_2 is the simplest example of a sequential space which is not Fréchet-Urysohn.

We recall that a topological space X is *sequential* if a subset $A \subset X$ is closed if and only if A is *sequentially closed* in the sense that A contains the limit point of any sequence $(a_n) \subset A$, convergent in X . A topological space X is *Fréchet-Urysohn* if for any cluster point $a \in X$ of a subset $A \subset X$ there is a sequence $(a_n) \subset A$, convergent to a .

Observe that $\aleph_0 = \text{cs}_\chi^*(S_2) = \text{cs}_\chi(S_2) = \text{sb}_\chi(S_2) < \chi(S_2) = \mathfrak{d}$ while $\aleph_0 = \text{cs}_\chi^*(S_\omega) = \text{cs}_\chi(S_\omega) < \text{sb}_\chi(S_\omega) = \chi(S_\omega) = \mathfrak{d}$. Here \mathfrak{d} is the well-known in Set Theory small uncountable cardinal equal to the cofinality of the partially ordered set \mathbb{N}^ω endowed with the natural partial order: $(x_n) \leq (y_n)$ iff $x_n \leq y_n$ for all n , see [Va]. Besides \mathfrak{d} , we will need two other small cardinals: \mathfrak{b} defined as the smallest size of a subset of uncountable cofinality in $(\mathbb{N}^\omega, \leq)$, and \mathfrak{p} equal to the smallest size $|\mathcal{F}|$ of a family of infinite subsets of ω closed under finite intersections and having no infinite pseudo-intersection in the sense that there is no infinite subset $I \subset \omega$ such that the complement $I \setminus F$ is finite for any $F \in \mathcal{F}$, see [Va], [vD]. It is known that $\aleph_1 \leq \mathfrak{p} \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$ where \mathfrak{c} stands for the size of continuum. Martin Axiom implies $\mathfrak{p} = \mathfrak{b} = \mathfrak{d} = \mathfrak{c}$, [MS]. On the other hand, for any uncountable regular

cardinals $\lambda \leq \kappa$ there is a model of ZFC with $\mathfrak{p} = \mathfrak{b} = \mathfrak{d} = \lambda$ and $\mathfrak{c} = \kappa$, see [vD, 5.1].

Unlike to the cardinal invariants cs_χ , sb_χ and χ which can be distinguished on simple spaces, the difference between the cardinal invariants cs_χ and cs_χ^* is more subtle: they cannot be distinguished in some models of Set Theory!

Proposition 2. *Let X be a topological space. Then $cs_\chi^*(X) = cs_\chi(X)$ provided one of the following conditions is satisfied:*

- (1) $cs_\chi^*(X) < \mathfrak{p}$;
- (2) $\kappa^{\aleph_0} \leq cs_\chi^*(X)$ for any cardinal $\kappa < cs_\chi^*(X)$;
- (3) $\mathfrak{p} = \mathfrak{c}$ and $\lambda^\omega \leq \kappa$ for any cardinals $\lambda < \kappa \geq \mathfrak{c}$;
- (4) $\mathfrak{p} = \mathfrak{c}$ (this is so under MA) and X is countable;
- (5) the Generalized Continuum Hypothesis holds.

Unlike to the usual character, the cs^* -, cs -, and sb -characters behave nicely with respect to many countable topological operations.

Among such operation there are: the Tychonov product, the box-product, producing a sequentially homeomorphic copy, taking image under a sequentially open map, and forming inductive topologies.

As usual, under *the box-product* $\prod_{i \in \mathcal{I}} X_i$ of topological spaces X_i , $i \in \mathcal{I}$, we understand the Cartesian product $\prod_{i \in \mathcal{I}} X_i$ endowed with the box-product topology generated by the base consisting of products $\prod_{i \in \mathcal{I}} U_i$ where each U_i is open in X_i . In contrast, by $\prod_{i \in \mathcal{I}} X_i$ we denote the usual Cartesian product of the spaces X_i , endowed with the Tychonov product topology.

We say that a topological space X carries *the inductive topology* with respect to a cover \mathcal{C} of X if a subset $F \subset X$ is closed in X if and only if the intersection $F \cap C$ is closed in C for each element $C \in \mathcal{C}$. For a cover \mathcal{C} of X let $\text{ord}(\mathcal{C}) = \sup_{x \in X} \text{ord}(\mathcal{C}, x)$ where $\text{ord}(\mathcal{C}, x) = |\{C \in \mathcal{C} : x \in C\}|$. A topological space X carrying the inductive topology with respect to a countable cover by closed metrizable (resp. compact, compact metrizable) subspaces is called an \mathcal{M}_ω -space (resp. a k_ω -space, \mathcal{MK}_ω -space).

A function $f : X \rightarrow Y$ between topological spaces is called *sequentially continuous* if for any convergent sequence (x_n) in X the sequence $(f(x_n))$ is convergent in Y to $f(\lim x_n)$; f is called a *sequential homeomorphism* if f is bijective and both f and f^{-1} are sequentially continuous. Topological spaces X, Y are defined to be *sequentially homeomorphic* if there is a sequential homeomorphism $h : X \rightarrow Y$. Observe that two spaces are sequentially homeomorphic if and only if their sequential coreflexions are homeomorphic. Under *the sequential coreflexion* σX of a topological space X we understand X endowed with the topology consisting of all sequentially open subsets of X (a subset U of X is *sequentially open* if its complement is sequentially closed in X ; equivalently U is a sequential barrier at each point $x \in U$). Note that the identity map $\text{id} : \sigma X \rightarrow X$ is continuous while its inverse is sequentially continuous, see [Lin].

A map $f : X \rightarrow Y$ is *sequentially open* if for any point $x_0 \in X$ and a sequence $S \subset Y$ convergent to $f(x_0)$ there is a sequence $T \subset X$ convergent to x_0 and such that $f(T) \subset S$. Observe that a bijective map f is sequentially open if its inverse f^{-1} is sequentially continuous.

The following technical Proposition is an easy consequence of the corresponding definitions.

- Proposition 3.** (1) *If X is a subspace of a topological space Y , then $cs_\chi^*(X) \leq cs_\chi^*(Y)$, $cs_\chi(X) \leq cs_\chi(Y)$ and $sb_\chi(X) \leq sb(Y)$.*
- (2) *If $f : X \rightarrow Y$ is a surjective continuous sequentially open map between topological spaces, then $cs_\chi^*(Y) \leq cs_\chi^*(X)$ and $sb_\chi(Y) \leq sb_\chi(X)$.*
- (3) *If $f : X \rightarrow Y$ is a surjective sequentially continuous sequentially open map between topological spaces, then $\min\{cs_\chi^*(Y), \aleph_1\} \leq \min\{cs_\chi^*(X), \aleph_1\}$, $\min\{cs_\chi(Y), \aleph_1\} \leq \min\{cs_\chi(X), \aleph_1\}$, and $\min\{sb_\chi(Y), \aleph_1\} \leq \min\{sb_\chi(X), \aleph_1\}$.*
- (4) *If X, Y are sequentially homeomorphic topological spaces, then $\min\{cs_\chi^*(X), \aleph_1\} = \min\{cs_\chi(X), \aleph_1\} = \min\{cs_\chi(Y), \aleph_1\} = \min\{cs_\chi^*(Y), \aleph_1\}$, and $\min\{sb_\chi(Y), \aleph_1\} = \min\{sb_\chi(X), \aleph_1\}$.*
- (5) *$\min\{sb_\chi(X), \aleph_1\} = \min\{sb_\chi(\sigma X), \aleph_1\} \leq sb_\chi(\sigma X) \geq sb_\chi(X)$ and $cs_\chi(X) \leq cs_\chi(\sigma X) \geq \min\{cs_\chi(\sigma X), \aleph_1\} = \min\{cs_\chi(X), \aleph_1\} = \min\{cs_\chi^*(X), \aleph_1\} = \min\{cs_\chi^*(\sigma X), \aleph_1\} \leq cs_\chi^*(\sigma X) \geq cs_\chi^*(X)$ for any topological space X .*
- (6) *If $X = \prod_{i \in \mathcal{I}} X_i$ is the Tychonov product of topological spaces X_i , $i \in \mathcal{I}$, then $cs_\chi^*(X) \leq \sum_{i \in \mathcal{I}} cs_\chi^*(X_i)$, $cs_\chi(X) \leq \sum_{i \in \mathcal{I}} cs_\chi(X_i)$ and $sb_\chi(X) \leq \sum_{i \in \mathcal{I}} sb_\chi(X_i)$.*
- (7) *If $X = \square_{i \in \mathcal{I}} X_i$ is the box-product of topological spaces X_i , $i \in \mathcal{I}$, then $cs_\chi^*(X) \leq \sum_{i \in \mathcal{I}} cs_\chi^*(X_i)$ and $cs_\chi(X) \leq \sum_{i \in \mathcal{I}} cs_\chi(X_i)$.*
- (8) *If a topological space X carries the inductive topology with respect to a cover \mathcal{C} of X , then $cs_\chi^*(X) \leq \text{ord}(\mathcal{C}) \cdot \sup_{C \in \mathcal{C}} cs_\chi^*(C)$.*
- (9) *If a topological space X carries the inductive topology with respect to a point-countable cover \mathcal{C} of X , then $cs_\chi(X) \leq \sup_{C \in \mathcal{C}} cs_\chi(C)$.*
- (10) *If a topological space X carries the inductive topology with respect to a point-finite cover \mathcal{C} of X , then $sb_\chi(X) \leq \sup_{C \in \mathcal{C}} sb_\chi(C)$.*

Since each first-countable space has countable cs^* -character, it is natural to consider the class of topological spaces with countable cs^* -character as a class of generalized metric spaces. However this class contains very non-metrizable spaces like $\beta\mathbb{N}$, the Stone-Ćech compactification of the discrete space of positive integers. The reason is that $\beta\mathbb{N}$ contains no non-trivial convergent sequence. To avoid such pathologies we shall restrict ourselves by sequential spaces. Observe that a topological space is sequential provided X carries the inductive topology with respect to a cover by sequential subspaces. In particular, each \mathcal{M}_ω -space is sequential and has countable cs^* -character. Our principal result states that for topological groups the converse is also true. Under an \mathcal{M}_ω -group (resp. \mathcal{MK}_ω -group) we understand a topological group whose underlying topological space is an \mathcal{M}_ω -space (resp. \mathcal{MK}_ω -space).

Theorem 1. *Each sequential topological group G with countable cs^* -character is an \mathcal{M}_ω -group. More precisely, either G is metrizable or else G contains an open \mathcal{MK}_ω -subgroup H and is homeomorphic to the product $H \times D$ for some discrete space D .*

For \mathcal{M}_ω -groups the second part of this theorem was proven in [Ba₁]. Theorem 1 has many interesting corollaries.

At first we show that for sequential topological groups with countable cs^* -character many important cardinal invariants are countable, coincide or take some fixed values. Let us remind some definitions, see [En₁]. For a topological space X recall that

- the *pseudocharacter* $\psi(X)$ is the smallest cardinal κ such that each one-point set $\{x\} \subset X$ can be written as the intersection $\{x\} = \bigcap \mathcal{U}$ of some family \mathcal{U} of open subsets of X with $|\mathcal{U}| \leq \kappa$;
- the *cellularity* $c(X)$ is the smallest cardinal κ such that X contains no family \mathcal{U} of size $|\mathcal{U}| > \kappa$ consisting of non-empty pairwise disjoint open subsets;
- the *Lindelöf number* $l(X)$ is the smallest cardinal κ such that each open cover of X contains a subcover of size $\leq \kappa$;
- the *density* $d(X)$ is the smallest size of a dense subset of X ;
- the *tightness* $t(X)$ is the smallest cardinal κ such that for any subset $A \subset X$ and a point $a \in \bar{A}$ from its closure there is a subset $B \subset A$ of size $|B| \leq \kappa$ with $a \in \bar{B}$;
- the *extent* $e(X)$ is the smallest cardinal κ such that X contains no closed discrete subspace of size $> \kappa$;
- the *compact covering number* $kc(X)$ is the smallest size of a cover of X by compact subsets;
- the *weight* $w(X)$ is the smallest size of a base of the topology of X ;
- the *network weight* $nw(X)$ is the smallest size $|\mathcal{N}|$ of a topological network for X (a family \mathcal{N} of subsets of X is a *topological network* if for any open set $U \subset X$ and any point $x \in U$ there is $N \in \mathcal{N}$ with $x \in N \subset U$);
- the *k-network weight* $knw(X)$ is the smallest size $|\mathcal{N}|$ of a *k-network* for X (a family \mathcal{N} of subsets of X is a *k-network* if for any open set $U \subset X$ and any compact subset $K \subset U$ there is a finite subfamily $\mathcal{M} \subset \mathcal{N}$ with $K \subset \bigcup \mathcal{M} \subset U$).

For each topological space X these cardinal invariants relate as follows:

$$\max\{c(X), l(X), e(X)\} \leq nw(X) \leq knw(X) \leq w(X).$$

For metrizable spaces all of them are equal, see [En₁, 4.1.15].

In the class of *k*-spaces there is another cardinal invariant, the *k*-ness introduced by E. van Douwen, see [vD, §8]. We remind that a topological space X is called a *k-space* if it carries the inductive topology with respect to the cover of X by all compact subsets. It is clear that each sequential space is a *k-space*. The *k-ness* $k(X)$ of a *k-space* is the smallest size $|\mathcal{K}|$ of a cover \mathcal{K} of X by compact subsets such that X carries the inductive topology with respect to the cover \mathcal{K} . It is interesting to notice that $k(\mathbb{N}^\omega) = \mathfrak{d}$ while $k(\mathbb{Q}) = \mathfrak{b}$, see [vD]. Proposition 3(8) implies that $cs_\chi^*(X) \leq k(X) \cdot \psi(X) \geq kc(X)$ for each *k-space* X . Observe also that a topological space X is a k_ω -space if and only if X is a *k-space* with $k(X) \leq \aleph_0$.

Besides cardinal invariants we shall consider an ordinal invariant, called the sequential order. Under the *sequential closure* $A^{(1)}$ of a subset A of a topological space X we understand the set of all limit point of sequences $(a_n) \subset A$, convergent in X . Given an ordinal α define the α -th sequential closure $A^{(\alpha)}$ of A by transfinite induction: $A^{(\alpha)} = \bigcup_{\beta < \alpha} (A^{(\beta)})^{(1)}$. Under the *sequential order* $so(X)$ of a topological space X we understand the smallest ordinal α such that $A^{(\alpha+1)} = A^{(\alpha)}$ for any subset $A \subset X$. Observe that a topological space X is Fréchet-Urysohn if and only if $so(X) \leq 1$; X is sequential if and only if $cl_X(A) = A^{(so(X))}$ for any subset $A \subset X$.

Besides purely topological invariants we shall also consider a cardinal invariant, specific for topological groups. For a topological group G let $ib(G)$, the *boundedness index* of G be the smallest cardinal κ such that for any nonempty open set $U \subset G$ there is a subset $F \subset G$ of size $|F| \leq \kappa$ such that $G = F \cdot U$. It is known that

$ib(G) \leq \min\{c(G), l(G), e(G)\}$ and $w(G) = ib(G) \cdot \chi(G)$ for each topological group, see [Tk].

Theorem 2. *Each sequential topological group G with countable cs^* -character has the following properties: $\psi(G) \leq \aleph_0$, $sb_\chi(G) = \chi(G) \in \{1, \aleph_0, \mathfrak{d}\}$, $ib(G) = c(G) = d(G) = l(G) = e(G) = nw(G) = knw(G)$, and $so(G) \in \{1, \omega_1\}$.*

We shall derive from Theorems 1 and 2 an unexpected metrization theorem for topological groups. But first we need to remind the definitions of some of α_i -spaces, $i = 1, \dots, 6$ introduced by A.V. Arhangel'ski in [Ar₂], [Ar₄]. We also define a wider class of α_7 -spaces.

A topological space X is called

- an α_1 -space if for any sequences $S_n \subset X$, $n \in \omega$, convergent to a point $x \in X$ there is a sequence $S \subset X$ convergent to x and such that $S_n \setminus S$ is finite for all n ;
- an α_4 -space if for any sequences $S_n \subset X$, $n \in \omega$, convergent to a point $x \in X$ there is a sequence $S \subset X$ convergent to x and such that $S_n \cap S \neq \emptyset$ for infinitely many sequences S_n ;
- an α_7 -space if for any sequences $S_n \subset X$, $n \in \omega$, convergent to a point $x \in X$ there is a sequence $S \subset X$ convergent to some point y of X and such that $S_n \cap S \neq \emptyset$ for infinitely many sequences S_n ;

Under a sequence converging to a point x of a topological space X we understand any countable infinite subset S of X such that $S \setminus U$ is finite for any neighborhood U of x . Each α_1 -space is α_4 and each α_4 -space is α_7 . Quite often α_7 -spaces are α_4 , see Lemma 7. Observe also that each sequentially compact space is α_7 . It can be shown that a topological space X is an α_7 -space if and only if it contains no closed copy of the sequential fan S_ω in its sequential coreflexion σX . If X is an α_4 -space, then σX contains no topological copy of S_ω .

We remind that a topological group G is *Weil complete* if it is complete in its left (equivalently, right) uniformity. According to [PZ, 4.1.6], each k_ω -group is Weil complete. The following metrization theorem can be easily derived from Theorems 1, 2 and elementary properties of \mathcal{MK}_ω -groups.

Theorem 3. *A sequential topological group G with countable cs^* -character is metrizable if one of the following conditions is satisfied:*

- (1) $so(G) < \omega_1$;
- (2) $sb_\chi(G) < \mathfrak{d}$;
- (3) $ib(G) < k(G)$;
- (4) G is Fréchet-Urysohn;
- (5) G is an α_7 -space;
- (6) G contains no closed copy of S_ω or S_2 ;
- (7) G is not Weil complete;
- (8) G is Baire;
- (9) $ib(G) < |G| < 2^{\aleph_0}$.

According to Theorem 1, each sequential topological group with countable cs^* -character is an \mathcal{M}_ω -group. The first author has proved in [Ba₃] that the topological structure of a non-metrizable punctiform \mathcal{M}_ω -group is completely determined by its density and the compact scatteredness rank.

Recall that a topological space X is *punctiform* if X contains no compact connected subspace containing more than one point, see [En₂, 1.4.3]. In particular, each zero-dimensional space is punctiform.

Next, we remind the definition of the scatteredness height. Given a topological space X let $X_{(1)} \subset X$ denote the set of all non-isolated points of X . For each ordinal α define the α -th derived set $X_{(\alpha)}$ of X by transfinite induction: $X_{(\alpha)} = \bigcap_{\beta < \alpha} (X_{(\beta)})_{(1)}$. Under the *scatteredness height* $\text{sch}(X)$ of X we understand the smallest ordinal α such that $X_{(\alpha+1)} = X_{(\alpha)}$. A topological space X is *scattered* if $X_{(\alpha)} = \emptyset$ for some ordinal α . Under the *compact scatteredness rank* of a topological space X we understand the ordinal $\text{scr}(X) = \sup\{\text{sch}(K) : K \text{ is a scattered compact subspace of } X\}$.

Theorem 4. *Two non-metrizable sequential punctiform topological groups G, H with countable cs^* -character are homeomorphic if and only if $d(G) = d(H)$ and $\text{scr}(G) = \text{scr}(H)$.*

This theorem follows from Theorem 1 and ‘‘Main Theorem’’ of [Ba₃] asserting that two non-metrizable punctiform \mathcal{M}_ω -groups G, H are homeomorphic if and only if $d(G) = d(H)$ and $\text{scr}(G) = \text{scr}(H)$. For countable k_ω -groups this fact was proven by E.Zelenyuk [Ze₁].

The topological classification of non-metrizable sequential locally convex spaces with countable cs^* -character is even more simple. Any such a space is homeomorphic either to \mathbb{R}^∞ or to $\mathbb{R}^\infty \times Q$ where $Q = [0, 1]^\omega$ is the Hilbert cube and \mathbb{R}^∞ is a linear space of countable algebraic dimension, carrying the strongest locally convex topology. It is well-known that this topology is inductive with respect to the cover of \mathbb{R}^∞ by finite-dimensional linear subspaces. The topological characterization of the spaces \mathbb{R}^∞ and $\mathbb{R}^\infty \times Q$ was given in [Sa]. In [Ba₂] it was shown that each infinite-dimensional locally convex \mathcal{MK}_ω -space is homeomorphic to \mathbb{R}^∞ or $\mathbb{R}^\infty \times Q$. This result together with Theorem 1 implies the following classification

Corollary 1. *Each non-metrizable sequential locally convex space with countable cs^* -character is homeomorphic to \mathbb{R}^∞ or $\mathbb{R}^\infty \times Q$.*

As we saw in Theorem 2, each sequential topological group with countable cs^* -character has countable pseudocharacter. The proof of this result is based on the fact that compact subsets of sequential topological groups with countable cs^* -character are first countable. This naturally leads to a conjecture that compact spaces with countable cs^* -character are first countable. Surprisingly, but this conjecture is false: assuming the Continuum Hypothesis N. Yakovlev [Ya] has constructed a scattered sequential compactum which has countable sb -character but fails to be first countable. In [Ny₂] P.Nyikos pointed out that the Yakovlev construction still can be carried under the assumption $\mathfrak{b} = \mathfrak{c}$. More precisely, we have

Proposition 4. *Under $\mathfrak{b} = \mathfrak{c}$ there is a regular locally compact locally countable space Y whose one-point compactification αY is sequential and satisfies $\aleph_0 = \text{sb}_\chi(\alpha Y) < \psi(\alpha Y) = \mathfrak{c}$.*

We shall use the above proposition to construct examples of topologically homogeneous spaces with countable cs -character and uncountable pseudocharacter. This shows that Theorem 2 is specific for topological groups and cannot be generalized to topologically homogeneous spaces. We remind that a topological space X

is *topologically homogeneous* if for any points $x, y \in X$ there is a homeomorphism $h : X \rightarrow X$ with $h(x) = y$.

Theorem 5.

- (1) *There is a topologically homogeneous countable regular k_ω -space X_1 with $\aleph_0 = \text{sb}_\chi(X_1) < \chi(X_1) = \mathfrak{d}$ and $\text{so}(X_1) = \omega$;*
- (2) *Under $\mathfrak{b} = \mathfrak{c}$ there is a sequential topologically homogeneous zero-dimensional k_ω -space X_2 with $\aleph_0 = \text{cs}_\chi(X_2) < \psi(X_2) = \mathfrak{c}$;*
- (3) *Under $\mathfrak{b} = \mathfrak{c}$ there is a sequential topologically homogeneous totally disconnected space X_3 with $\aleph_0 = \text{sb}_\chi(X_3) < \psi(X_3) = \mathfrak{c}$.*

We remind that a space X is *totally disconnected* if for any distinct points $x, y \in X$ there is a continuous function $f : X \rightarrow \{0, 1\}$ such that $f(x) \neq f(y)$, see [En₂].

Remark 1. The space X_1 from Theorem 5(1) is the well-known Arhangel'ski-Franklin example [AF] (see also [Co, 10.1]) of a countable topologically homogeneous k_ω -space, homeomorphic to no topological group (this also follows from Theorem 2). On the other hand, according to [Ze₂], each topologically homogeneous countable regular space (in particular, X_1) is homeomorphic to a *quasitopological group*, that is a topological space endowed with a separately continuous group operation with continuous inversion. This shows that Theorem 2 cannot be generalized onto quasitopological groups (see however [Zd] for generalizations of Theorems 1 and 2 to some other topological-algebraic structures).

Next, we find conditions under which a space with countable cs^* -character is first-countable or has countable sb -character. Following [Ar₃] we define a topological space X to be *c-sequential* if for each closed subspace $Y \subset X$ and each non-isolated point y of Y there is a sequence $(y_n) \subset Y \setminus \{y\}$ convergent to y . It is clear that each sequential space is *c-sequential*. A point x of a topological space X is called *regular G_δ* if $\{x\} = \bigcap \mathcal{B}$ for some countable family \mathcal{B} of closed neighborhoods of x in X , see [Lin].

First we characterize spaces with countable sb -character (the first three items of this characterization were proved by Lin [Lin, 3.13] in terms of (universally) *csf*-countable spaces).

Proposition 5. *For a Hausdorff space X the following conditions are equivalent:*

- (1) *X has countable sb -character;*
- (2) *X is an α_1 -space with countable cs^* -character;*
- (3) *X is an α_4 -space with countable cs^* -character;*
- (4) *$\text{cs}_\chi^*(X) \leq \aleph_0$ and $\text{sb}_\chi(X) < \mathfrak{p}$.*

Moreover, if X is *c-sequential* and each point of X is *regular G_δ* , then the conditions

- (1)–(4) *are equivalent to:*
- (5) *$\text{cs}_\chi^*(X) \leq \aleph_0$ and $\text{sb}_\chi(X) < \mathfrak{d}$.*

Next, we give a characterization of first-countable spaces in the same spirit (the equivalences (1) \Leftrightarrow (2) \Leftrightarrow (5) were proved by Lin [Lin, 2.8]).

Proposition 6. *For a Hausdorff space X with countable cs^* -character the following conditions are equivalent:*

- (1) *X is first-countable;*
- (2) *X is Fréchet-Urysohn and has countable sb -character;*
- (3) *X is Fréchet-Urysohn α_7 -space;*

(4) $\chi(X) < \mathfrak{p}$ and X has countable tightness.

Moreover, if each point of X is regular G_δ , then the conditions (1)–(4) are equivalent to:

- (5) X is a sequential space containing no closed copy of S_2 or S_ω ;
(6) X is a sequential space with $\chi(X) < \mathfrak{d}$.

For Fréchet-Urysohn (resp. dyadic) compacta the countability of the cs^* -character is equivalent to the first countability (resp. the metrizability). We remind that a compact Hausdorff space X is called *dyadic* if X is a continuous image of the Cantor discontinuum $\{0, 1\}^\kappa$ for some cardinal κ .

Proposition 7.

- (1) A Fréchet-Urysohn countably compact space is first-countable if and only if it has countable cs^* -character.
(2) A dyadic compactum is metrizable if and only if it has countable cs^* -character.

In light of Proposition 7(1) one can suggest that $cs_\chi^*(X) = \chi(X)$ for any compact Fréchet-Urysohn space X . However that is not true: under CH, $cs_\chi(\alpha D) \neq \chi(\alpha D)$ for the one-point compactification αD of a discrete space D of size $|D| = \aleph_2$. Surprisingly, but the problem of calculating the cs^* - and cs -characters of the spaces αD is not trivial and the definitive answer is known only under the Generalized Continuum Hypothesis. First we note that the cardinals $cs_\chi^*(\alpha D)$ and $cs_\chi(\alpha D)$ admit an interesting interpretation which will be used for their calculation.

Proposition 8. *Let D be an infinite discrete space. Then*

- (1) $cs_\chi^*(\alpha D) = \min\{w(X) : X \text{ is a (regular zero-dimensional) topological space of size } |X| = |D| \text{ containing non no-trivial convergent sequence}\}$;
(2) $cs_\chi(\alpha D) = \min\{w(X) : X \text{ is a (regular zero-dimensional) topological space of size } |X| = |D| \text{ containing no countable non-discrete subspace}\}$.

For a cardinal κ we put $\log \kappa = \min\{\lambda : \kappa \leq 2^\lambda\}$ and $\text{cof}([\kappa]^{\leq \omega})$ be the smallest size of a collection $\mathcal{C} \subset [\kappa]^{\leq \omega}$ such that each at most countable subset $S \subset \kappa$ lies in some element $C \in \mathcal{C}$. Observe that $\text{cof}([\kappa]^{\leq \omega}) \leq \kappa^\omega$ but sometimes the inequality can be strict: $1 = \text{cof}([\aleph_0]^{\leq \omega}) < \aleph_0$ and $\aleph_1 = \text{cof}([\aleph_1]^{\leq \omega}) < \aleph_1^{\aleph_0}$. In the following proposition we collect all the information on the cardinals $cs_\chi^*(\alpha D)$ and $cs_\chi(\alpha D)$ we know.

Proposition 9. *Let D be an uncountable discrete space. Then*

- (1) $\aleph_1 \cdot \log |D| \leq cs_\chi^*(\alpha D) \leq cs_\chi(\alpha D) \leq \min\{|D|, 2^{\aleph_0} \cdot \text{cof}([\log |D|]^{\leq \omega})\}$ while $sb_\chi(\alpha D) = \chi(\alpha D) = |D|$;
(2) $cs_\chi^*(\alpha D) = cs_\chi(\alpha D) = \aleph_1 \cdot \log |D|$ under GCH.

In spite of numerous efforts some annoying problems concerning cs^* - and cs -characters still rest open.

Problem 1. *Is there a (necessarily consistent) example of a space X with $cs_\chi^*(X) \neq cs_\chi(X)$? In particular, is $cs_\chi^*(\alpha \mathfrak{c}) \neq cs_\chi(\alpha \mathfrak{c})$ in some model of ZFC?*

In light of Proposition 8 it is natural to consider the following three cardinal characteristics of the continuum which seem to be new:

$\mathfrak{w}_1 = \min\{w(X) : X \text{ is a topological space of size } |X| = \mathfrak{c} \text{ containing no non-trivial convergent sequence}\};$

$\mathfrak{w}_2 = \min\{w(X) : X \text{ is a topological space of size } |X| = \mathfrak{c} \text{ containing no non-discrete countable subspace}\};$

$\mathfrak{w}_3 = \min\{w(X) : X \text{ is a } P\text{-space of size } |X| = \mathfrak{c}\}.$

As expected, a P -space is a T_1 -space whose any G_δ -subset is open. Observe that $\mathfrak{w}_1 = \text{cs}_\chi^*(\alpha\mathfrak{c})$ while $\mathfrak{w}_2 = \text{cs}_\chi(\alpha\mathfrak{c})$. It is clear that $\aleph_1 \leq \mathfrak{w}_1 \leq \mathfrak{w}_2 \leq \mathfrak{w}_3 \leq \mathfrak{c}$ and hence the cardinals \mathfrak{w}_i , $i = 1, 2, 3$, fall into the category of small uncountable cardinals, see [Va].

Problem 2. *Are the cardinals \mathfrak{w}_i , $i = 1, 2, 3$, equal to (or can be estimated via) some known small uncountable cardinals considered in Set Theory? Is $\mathfrak{w}_1 < \mathfrak{w}_2 < \mathfrak{w}_3$ in some model of ZFC?*

Our next question concerns the assumption $\mathfrak{b} = \mathfrak{c}$ in Theorem 5.

Problem 3. *Is there a ZFC-example of a sequential space X with $\text{sb}_\chi(X) < \psi(X)$ or at least $\text{cs}_\chi^*(X) < \psi(X)$?*

Propositions 1 and 5 imply that $\text{sb}_\chi(X) \in \{1, \aleph_0\} \cup [\mathfrak{d}, \mathfrak{c}]$ for any c -sequential topological space X with countable cs^* -character. On the other hand, for a sequential topological group G with countable cs^* -character we have a more precise estimate $\text{sb}_\chi(G) \in \{1, \aleph_0, \mathfrak{d}\}$.

Problem 4. *Is $\text{sb}_\chi(X) \in \{1, \aleph_0, \mathfrak{d}\}$ for any sequential space X with countable cs^* -character?*

As we saw in Proposition 7, $\chi(X) \leq \aleph_0$ for any Fréchet-Urysohn compactum X with $\text{cs}_\chi(X) \leq \aleph_0$.

Problem 5. *Is $\text{sb}_\chi(X) \leq \aleph_0$ for any sequential (scattered) compactum X with $\text{cs}_\chi(X) \leq \aleph_0$?*

Now we pass to proofs of our results.

ON SEQUENCE TREES IN TOPOLOGICAL GROUPS

Our basic instrument in proofs of main results is the concept of a sequence tree. As usual, under a *tree* we understand a partially ordered subset (T, \leq) such that for each $t \in T$ the set $\downarrow t = \{\tau \in T : \sigma \leq t\}$ is well-ordered by the order \leq . Given an element $t \in T$ let $\uparrow t = \{\tau \in T : \tau \geq t\}$ and $\text{succ}(t) = \min(\uparrow t \setminus \{t\})$ be the set of successors of t in T . A maximal linearly ordered subset of a tree T is called a *branch* of T . By $\max T$ we denote the set of maximal elements of the tree T .

Definition 1. Under a *sequence tree* in a topological space X we understand a tree (T, \leq) such that

- $T \subset X$;
- T has no infinite branch;
- for each $t \notin \max T$ the set $\min(\uparrow t \setminus \{t\})$ of successors of t is countable and converges to t .

Saying that a subset S of a topological space X converges to a point $t \in X$ we mean that for each neighborhood $U \subset X$ of t the set $S \setminus U$ is finite.

The following lemma is well-known and can be easily proven by transfinite induction (on the ordinal $s(a, A) = \min\{\alpha : a \in A^{(\alpha)}\}$ for a subset A of a sequential space and a point $a \in \bar{A}$ from its closure)

Lemma 1. *A point $a \in X$ of a sequential topological space X belongs to the closure of a subset $A \subset X$ if and only if there is a sequence tree $T \subset X$ with $\min T = \{a\}$ and $\max T \subset A$.*

For subsets A, B of a group G let $A^{-1} = \{x^{-1} : x \in A\} \subset G$ be the inversion of A in G and $AB = \{xy : x \in A, y \in B\} \subset G$ be the product of A, B in G . The following two lemmas will be used in the proof of Theorem 1.

Lemma 2. *A sequential subspace $F \subset X$ of a topological group G is first countable if the subspace $F^{-1}F \subset G$ has countable sb-character at the unit e of the group G .*

Proof. Our proof starts with the observation that it is sufficient to consider the case $e \in F$ and prove that F has countable character at e .

Let $\{S_n : n \in \omega\}$ be a decreasing sb-network at e in $F^{-1}F$. First we show that for every $n \in \omega$ there exists $m > n$ such that $S_m^2 \cap (F^{-1}F) \subset S_n$. Otherwise, for every $m \in \omega$ there would exist $x_m, y_m \in S_m$ with $x_m y_m \in (F^{-1}F) \setminus S_n$. Taking into account that $\lim_{m \rightarrow \infty} x_m = \lim_{m \rightarrow \infty} y_m = e$, we get $\lim_{m \rightarrow \infty} x_m y_m = e$. Since S_n is a sequential barrier at e , there is a number m with $x_m y_m \in S_n$, which contradicts to the choice of the points x_m, y_m .

Now let us show that for all $n \in \omega$ the set $S_n \cap F$ is a neighborhood of e in F . Suppose, conversely, that $e \in \text{cl}_F(F \setminus S_{n_0})$ for some $n_0 \in \omega$.

By Lemma 1 there exists a sequence tree $T \subset F$, $\min T = \{e\}$ and $\max T \subset F \setminus S_{n_0}$. To get a contradiction we shall construct an infinite branch of T . Put $x_0 = e$ and let m_0 be the smallest integer such that $S_{m_0}^2 \cap F^{-1}F \subset S_{n_0}$.

By induction, for every $i \geq 1$ find a number $m_i > m_{i-1}$ with $S_{m_i}^2 \cap F^{-1}F \subset S_{m_{i-1}}$ and a point $x_i \in \text{succ}(x_{i-1}) \cap (x_{i-1} S_{m_i})$. To show that such a choice is always possible, it suffices to verify that $x_{i-1} \notin \max T$. It follows from the inductive construction that $x_{i-1} \in F \cap (S_{m_0} \cdots S_{m_{i-1}}) \subset F \cap S_{m_0}^2 \subset S_{n_0}$ and thus $x_{i-1} \notin \max T$ because $\max T \subset F \setminus S_{n_0}$.

Therefore we have constructed an infinite branch $\{x_i : i \in \omega\}$ of the sequence tree T which is not possible. This contradiction finishes the proof. \square

Lemma 3. *A sequential α_7 -subspace F of a topological group G has countable sb-character provided the subspace $F^{-1}F \subset G$ has countable cs -character at the unit e of G .*

Proof. Suppose that $F \subset G$ is a sequential α_7 -space with $cs_\chi(F^{-1}F, e) \leq \aleph_0$. We have to prove that $sb_\chi(F, x) \leq \aleph_0$ for any point $x \in F$. Replacing F by Fx^{-1} , if necessary, we can assume that $x = e$ is the unit of the group G . Fix a countable family \mathcal{A} of subsets of G closed under group products in G , finite unions and finite intersections, and such that $F^{-1}F \in \mathcal{A}$ and $\mathcal{A}|F^{-1}F = \{A \cap (F^{-1}F) : A \in \mathcal{A}\}$ is a cs -network at e in $F^{-1}F$. We claim that the collection $\mathcal{A}|F = \{A \cap F : A \in \mathcal{A}\}$ is a sb-network at e in F .

Assuming the converse, we would find an open neighborhood $U \subset G$ of e such that for any element $A \in \mathcal{A}$ with $A \cap F \subset U$ the set $A \cap F$ fails to be a sequential barrier at e in F .

Let $\mathcal{A}' = \{A \in \mathcal{A} : A \subset F \cap U\} = \{A_n : n \in \omega\}$ and $B_n = \bigcup_{k \leq n} A_k$. Let $m_{-1} = 0$ and $U_{-1} \subset U$ be any closed neighborhood of e in G . By induction, for every $k \in \omega$ find a number $m_k > m_{k-1}$, a closed neighborhood $U_k \subset U_{k-1}$ of e in G , and a sequence $(x_{k,i})_{i \in \omega}$ convergent to e so that the following conditions are satisfied:

- (i) $\{x_{k,i} : i \in \omega\} \subset U_{k-1} \cap F \setminus B_{m_{k-1}}$;
- (ii) the set $F_k = \{x_{n,i} : n \leq k, i \in \omega\} \setminus B_{m_k}$ is finite;
- (iii) $U_k \cap (F_k \cup \{x_{i,j} : i, j \leq k\}) = \emptyset$ and $U_k^2 \subset U_{k-1}$.

The last condition implies that $U_0 U_1 \cdots U_k \subset U$ for every $k \geq 0$.

Consider the subspace $X = \{x_{k,i} : k, i \in \omega\}$ of F and observe that it is discrete (in itself). Denote by \bar{X} the closure of X in F and observe that $\bar{X} \setminus X$ is closed in F . We claim that e is an isolated point of $\bar{X} \setminus X$. Assuming the converse and applying Lemma 1 we would find a sequence tree $T \subset \bar{X}$ such that $\min T = \{e\}$, $\max T \subset X$, and $\text{succ}(e) \subset \bar{X} \setminus X$.

By induction, construct a (finite) branch $(t_i)_{i \leq n+1}$ of the tree T and a sequence $\{C_i : i \leq n\}$ of elements of the family \mathcal{A} such that $t_0 = e$, $|\text{succ}(t_i) \setminus t_i C_i| < \aleph_0$ and $C_i \subset U_i \cap (F^{-1}F)$, $t_{i+1} \in \text{succ}(t_i) \cap t_i C_i$, for each $i \leq n$. Note that the infinite set $\sigma = \text{succ}(t_n) \cap t_n C_n \subset X$ converges to the point $t_n \neq e$.

On the other hand, $\sigma \subset t_n C_n \subset t_{n-1} C_{n-1} C_n \subset \cdots \subset t_0 C_0 \cdots C_n \subset U_0 \cdots U_n \subset U$. It follows from our assumption on \mathcal{A} that $C_0 \cdots C_n \in \mathcal{A}$ and thus $(C_0 \cdots C_n) \cap F \subset B_{m_k}$ for some k . Consequently, $\sigma \subset X \cap B_{m_k}$ and $\sigma \subset \{x_{j,i} : j \leq k, i \in \omega\}$ by the item (i) of the construction of X . Since e is a unique cluster point of the set $\{x_{j,i} : j \leq k, i \in \omega\}$, the sequence σ cannot converge to $t_n \neq e$, which is a contradiction.

Thus e is an isolated point of $\bar{X} \setminus X$ and consequently, there is a closed neighborhood W of e in G such that the set $V = (\{e\} \cup X) \cap W$ is closed in F .

For every $n \in \omega$ consider the sequence $S_n = W \cap \{x_{n,i} : i \in \omega\}$ convergent to e . Since F is an α_7 -space, there is a convergent sequence $S \subset F$ such that $S \cap S_n \neq \emptyset$ for infinitely many sequences S_n . Taking into account that V is a closed subspace of F with $|V \cap S| = \aleph_0$, we conclude that the limit point $\lim S$ of S belongs to the set V . Moreover, we can assume that $S \subset V$. Since the space X is discrete, $\lim S \in V \setminus X = \{e\}$. Thus the sequence S converges to e . Since \mathcal{A}' is a cs-network at e in F , there is a number $n \in \omega$ such that A_n contains almost all members of the sequence S . Since $S_m \cap (S_k \cup A_n) = \emptyset$ for $m > k \geq n$, the sequence S cannot meet infinitely many sequences S_m . But this contradicts to the choice of S . \square

Following [vD, §8] by \mathbb{L} we denote the countable subspace of the plane \mathbb{R}^2 :

$$\mathbb{L} = \{(0, 0), (\frac{1}{n}, \frac{1}{nm}) : n, m \in \mathbb{N}\} \subset \mathbb{R}^2.$$

The space \mathbb{L} is locally compact at each point except for $(0, 0)$. Moreover, according to Lemma 8.3 of [vD], a first countable space X contains a closed topological copy of the space \mathbb{L} if and only if X is not locally compact.

The following important lemma was proven in [Ba₁] for normal sequential groups.

Lemma 4. *If a sequential topological group G contains a closed copy of the space \mathbb{L} , then G is an α_7 -space.*

Proof. Let $h : \mathbb{L} \rightarrow G$ be a closed embedding and let $x_0 = h(0, 0)$, $x_{n,m} = h(\frac{1}{n}, \frac{1}{nm})$ for $n, m \in \mathbb{N}$. To show that G is an α_7 -space, for every $n \in \mathbb{N}$ fix a sequence

$(y_{n,m})_{m \in \mathbb{N}} \subset G$, convergent to the unit e of G . Denote by $*$: $G \times G \rightarrow G$ the group operation on G .

It is easy to verify that for every n the subspace $D_n = \{x_{n,m} * y_{n,m} : m \in \mathbb{N}\}$ is closed and discrete in G . Hence there exists $k_n \in \mathbb{N}$ such that $x_0 \neq x_{n,m} * y_{n,m}$ for all $m > k_n$. Consider the subset

$$A = \{x_{n,m} * y_{n,m} : n > 0, m > k_n\}$$

and using the continuity of the group operation, show that $x_0 \notin A$ is a cluster point of A in G . Consequently, the set A is not closed and by the sequentiality of G , there is a sequence $S \subset A$ convergent to a point $a \notin A$. Since every space D_n is closed and discrete in G , we may replace S by a subsequence, and assume that $|S \cap D_n| \leq 1$ for every $n \in \mathbb{N}$. Consequently, S can be written as $S = \{x_{n_i, m_i} * y_{n_i, m_i} : i \in \omega\}$ for some number sequences (m_i) and (n_i) with $n_{i+1} > n_i$ for all i . It follows that the sequence $(x_{n_i, m_i})_{i \in \omega}$ converges to x_0 and consequently, the sequence $T = \{y_{n_i, m_i}\}_{i \in \omega}$ converges to $x_0^{-1} * a$. Since $T \cap \{y_{n_i, m_i}\}_{m \in \mathbb{N}} \neq \emptyset$ for every i , we conclude that G is an α_7 -space. \square

Lemma 4 allows us to prove the following unexpected

Lemma 5. *A non-metrizable sequential topological group G with countable cs -character has a countable cs -network at the unit, consisting of closed countably compact subsets of G .*

Proof. Given a non-metrizable sequential group G with countable cs -character we can apply Lemmas 2–4 to conclude that G contains no closed copy of the space \mathbb{L} . Fix a countable cs -network \mathcal{N} at e , closed under finite intersections and consisting of closed subspaces of G . We claim that the collection $\mathcal{C} \subset \mathcal{N}$ of all countably compact subsets $N \in \mathcal{N}$ forms a cs -network at e in G .

To show this, fix a neighborhood $U \subset G$ of e and a sequence $(x_n) \subset G$ convergent to e . We must find a countably compact set $M \in \mathcal{C}$ with $M \subset U$, containing almost all points x_n . Let $\mathcal{A} = \{A_k : k \in \omega\}$ be the collection of all elements $N \subset U$ of \mathcal{N} containing almost all points x_n . Now it suffices to find a number $n \in \omega$ such that the intersection $M = \bigcap_{k \leq n} A_k$ is countably compact. Suppose to the contrary, that for every $n \in \omega$ the set $\bigcap_{k \leq n} A_k$ is not countably compact. Then there exists a countable closed discrete subspace $K_0 \subset A_0$ with $K_0 \not\ni e$. Fix a neighborhood W_0 of e with $W_0 \cap K_0 = \emptyset$. Since \mathcal{N} is a cs -network at e , there exists $k_1 \in \omega$ such that $A_{k_1} \subset W_0$.

It follows from our hypothesis that there is a countable closed discrete subspace $K_1 \subset \bigcap_{k \leq k_1} A_k$ with $K_1 \ni e$. Proceeding in this fashion we construct by induction an increasing number sequence $(k_n)_{n \in \omega} \subset \omega$, a sequence $(K_n)_{n \in \omega}$ of countable closed discrete subspaces of G , and a sequence $(W_n)_{n \in \omega}$ of open neighborhoods of e such that $K_n \subset \bigcap_{k \leq k_n} A_k$, $W_n \cap K_n = \emptyset$, and $A_{k_{n+1}} \subset W_n$ for all $n \in \omega$.

It follows from the above construction that $\{e\} \cup \bigcup_{n \in \omega} K_n$ is a closed copy of the space \mathbb{L} which is impossible. \square

PROOFS OF MAIN RESULTS

Proof of Proposition 1. The first three items can be easily derived from the corresponding definitions. To prove the fourth item observe that for any cs^* -network \mathcal{N} at a point x of a topological space X , the family $\mathcal{N}' = \{\cup \mathcal{F} : \mathcal{F} \subset \mathcal{N}\}$ is an sb -network at x .

The proof of fifth item is more tricky. Fix any cs^* -network \mathcal{N} at a point $x \in X$ with $|\mathcal{N}| \leq \text{cs}_\chi^*(X)$. Let $\lambda = \text{cof}(|\mathcal{N}|)$ be the cofinality of the cardinal $|\mathcal{N}|$ and write $\mathcal{N} = \bigcup_{\alpha < \lambda} \mathcal{N}_\alpha$ where $\mathcal{N}_\alpha \subset \mathcal{N}_\beta$ and $|\mathcal{N}_\alpha| < |\mathcal{N}|$ for any ordinals $\alpha \leq \beta < \lambda$. Consider the family $\mathcal{M} = \{\cup \mathcal{C} : \mathcal{C} \in [\mathcal{N}_\alpha]^{\leq \omega}, \alpha < \lambda\}$ and observe that $|\mathcal{M}| \leq \lambda \cdot \sup\{[|\mathcal{N}_\alpha|^{\leq \omega}] : \alpha < \lambda\}$ where $[\kappa]^{\leq \omega} = \{A \subset \kappa : |A| \leq \aleph_0\}$. It rests to verify that \mathcal{M} is a cs -network at x .

Fix a neighborhood $U \subset X$ of x and a sequence $S \subset X$ convergent to x . For every $\alpha < \lambda$ choose a countable subset $\mathcal{C}_\alpha \subset \mathcal{N}_\alpha$ such that $\cup \mathcal{C}_\alpha \subset U$ and $S \cap (\cup \mathcal{C}_\alpha) = S \cap (\cup\{N \in \mathcal{N}_\alpha : N \subset U\})$. It follows that $\cup \mathcal{C}_\alpha \in \mathcal{M}$. Let $S_\alpha = S \cap (\cup \mathcal{C}_\alpha)$ and observe that $S_\alpha \subset S_\beta$ for $\alpha \leq \beta < \lambda$. To finish the proof it suffices to show that $S \setminus S_\alpha$ is finite for some $\alpha < \lambda$. Then the element $\cup \mathcal{C}_\alpha \subset U$ of \mathcal{M} will contain almost all members of the sequence S .

Separately, we shall consider the cases of countable and uncountable λ . If λ is uncountable, then it has uncountable cofinality and consequently, the transfinite sequence $(S_\alpha)_{\alpha < \lambda}$ eventually stabilizes, i.e., there is an ordinal $\alpha < \lambda$ such that $S_\beta = S_\alpha$ for all $\beta \geq \alpha$. We claim that the set $S \setminus S_\alpha$ is finite. Otherwise, $S \setminus S_\alpha$ would be a sequence convergent to x and there would exist an element $N \in \mathcal{N}$ with $N \subset U$ and infinite intersection $N \cap (S \setminus S_\alpha)$. Find now an ordinal $\beta \geq \alpha$ with $N \in \mathcal{N}_\beta$ and observe that $S \cap N \subset S_\beta = S_\alpha$ which contradicts to the choice of N .

If λ is countable and $S \setminus S_\alpha$ is infinite for any $\alpha < \lambda$, then we can find an infinite pseudo-intersection $T \subset S$ of the decreasing sequence $\{S \setminus S_\alpha\}_{\alpha < \lambda}$. Note that $T \cap S_\alpha$ is finite for every $\alpha < \lambda$. Since sequence T converges to x , there is an element $N \in \mathcal{N}$ such that $N \subset U$ and $N \cap T$ is infinite. Find $\alpha < \lambda$ with $N \in \mathcal{N}_\alpha$ and observe that $N \cap S \subset S_\alpha$. Then $N \cap T \subset N \cap T \cap S_\alpha \subset T \cap S_\alpha$ is finite, which contradicts to the choice of N .

Proof of Proposition 2. Let X be a topological space and fix a point $x \in X$.

(1) Suppose that $\text{cs}_\chi^*(X) < \mathfrak{p}$ and fix a cs^* -network \mathcal{N} at the point x such that $|\mathcal{N}| < \mathfrak{p}$. Without loss of generality, we can assume that the family \mathcal{N} is closed under finite unions. We claim that \mathcal{N} is a cs -network at x . Assuming the converse we would find a neighborhood $U \subset X$ of x and a sequence $S \subset X$ convergent to x such that $S \setminus N$ is infinite for any element $N \in \mathcal{N}$ with $N \subset U$. Since \mathcal{N} is closed under finite unions, the family $\mathcal{F} = \{S \setminus N : N \in \mathcal{N}, N \subset U\}$ is closed under finite intersections. Since $|\mathcal{F}| \leq |\mathcal{N}| < \mathfrak{p}$, the family \mathcal{F} has an infinite pseudo-intersection $T \subset S$. Consequently, $T \cap N$ is finite for any $N \in \mathcal{N}$ with $N \subset U$. But this contradicts to the facts that T converges to x and \mathcal{N} is a cs^* -network at x .

The items (2) and (3) follow from Propositions 1(5) and 2(1). The item (4) follows from (1,2) and the inequality $\chi(X) \leq \mathfrak{c}$ holding for any countable topological space X .

Finally, to derive (5) from (3) use the well-known fact that under GCH, $\lambda^{\aleph_0} \leq \kappa$ for any infinite cardinals $\lambda < \kappa$, see [HJ, 9.3.8].

Proof of Theorem 1. Suppose that G is a non-metrizable sequential group with countable cs^* -character. By Proposition 2(1), $\text{cs}_\chi(G) = \text{cs}_\chi^*(G) \leq \aleph_0$.

First we show that each countably compact subspace K of G is first-countable. The space K , being countably compact in the sequential space G , is sequentially compact and so are the sets $K^{-1}K$ and $(K^{-1}K)^{-1}(K^{-1}K)$ in G . The sequential compactness of $K^{-1}K$ implies that it is an α_7 -space. Since $\text{cs}_\chi((K^{-1}K)^{-1}(K^{-1}K)) \leq$

$cs_\chi(G) \leq \aleph_0$ we may apply Lemmas 3 and 2 to conclude that the space $K^{-1}K$ has countable sb -character and K has countable character.

Next, we show that G contains an open \mathcal{MK}_ω -subgroup. By Lemma 5, G has a countable cs -network \mathcal{K} consisting of countably compact subsets. Since the group product of two countably compact subspaces in G is countably compact, we may assume that \mathcal{K} is closed under finite group products in G . We can also assume that \mathcal{K} is closed under the inversion, i.e. $K^{-1} \in \mathcal{K}$ for any $K \in \mathcal{K}$. Then $H = \cup K$ is a subgroup of G . It follows that this subgroup is a sequential barrier at each of its points, and thus is open-and-closed in G . We claim that the topology on H is inductive with respect to the cover \mathcal{K} . Indeed, consider some $U \subset H$ such that $U \cap K$ is open in K for every $K \in \mathcal{K}$. Assuming that U is not open in H and using the sequentiality of H , we would find a point $x \in U$ and a sequence $(x_n)_{n \in \omega} \subset H \setminus U$ convergent to x . It follows that there are elements $K_1, K_2 \in \mathcal{K}$ such that $x \in K_1$ and K_2 contains almost all members of the sequence $(x^{-1}x_n)$. Then the product $K = K_1K_2$ contains almost all x_n and the set $U \cap K$, being an open neighborhood of x in K , contains almost all members of the sequence (x_n) , which is a contradiction.

As it was proved before each $K \in \mathcal{K}$ is first-countable, and consequently H has countable pseudocharacter, being the countable union of first countable subspaces. Then H admits a continuous metric. Since any continuous metric on a countably compact space generates its original topology, every $K \in \mathcal{K}$ is a metrizable compactum, and consequently H is an \mathcal{MK}_ω -subgroup of G .

Since H is an open subgroup of G , G is homeomorphic to $H \times D$ for some discrete space D .

Proof of Theorem 2. Suppose G is a non-metrizable sequential topological group with countable cs^* -character. By Theorem 1, G contains an open \mathcal{MK}_ω -subgroup H and is homeomorphic to the product $H \times D$ for some discrete space D . This implies that G has point-countable k -network. By a result of Shibakov [Shi], each sequential topological group with point-countable k -network and sequential order $< \omega_1$ is metrizable. Consequently, $so(G) = \omega_1$. It is clear that $\psi(G) = \psi(H) \leq \aleph_0$, $\chi(G) = \chi(H)$, $sb_\chi(G) = sb_\chi(H)$ and $ib(G) = c(G) = d(G) = l(G) = e(G) = nw(G) = knw(G) = |D| \cdot \aleph_0$.

To finish the proof it rests to show that $sb_\chi(H) = \chi(H) = \mathfrak{d}$. It follows from Lemmas 2 and 3 that the group H , being non-metrizable, is not α_7 and thus contains a copy of the sequential fan S_ω . Then $\mathfrak{d} = \chi(S_\omega) = sb_\chi(S_\omega) \leq sb_\chi(H) \leq \chi(H)$. To prove that $\chi(H) \leq \mathfrak{d}$ we shall apply a result of K. Sakai [Sa] asserting that the space $\mathbb{R}^\infty \times Q$ contains a closed topological copy of each \mathcal{MK}_ω -space and the well-known equality $\chi(\mathbb{R}^\infty \times Q) = \chi(\mathbb{R}^\infty) = \mathfrak{d}$ (following from the fact that \mathbb{R}^∞ carries the box-product topology, see [Sch, Ch.II, Ex.12]).

Proof of Theorem 5. First we describe two general constructions producing topologically homogeneous sequential spaces. For a locally compact space Z let $\alpha Z = Z \cup \{\infty\}$ be the one-point extension of Z endowed with the topology whose neighborhood base at ∞ consists of the sets $\alpha Z \setminus K$ where K is a compact subset of Z . Thus for a non-compact locally compact space Z the space αZ is nothing else but the one-point compactification of Z . Denote by $2^\omega = \{0, 1\}^\omega$ the Cantor cube.

Consider the subsets

$$\Xi(Z) = \{(c, (z_i)_{i \in \omega}) \in 2^\omega \times (\alpha Z)^\omega : z_i = \infty \text{ for all but finitely many indices } i\} \text{ and}$$

$$\Theta(Z) = \{(c, (z_i)_{i \in \omega}) \in 2^\omega \times (\alpha Z)^\omega : \exists n \in \omega \text{ such that } z_i \neq \infty \text{ if and only if } i < n\}.$$

Observe that $\Theta(Z) \subset \Xi(Z)$.

Endow the set $\Xi(Z)$ (resp. $\Theta(Z)$) with the strongest topology generating the Tychonov product topology on each compact subset from the family \mathcal{K}_Ξ (resp. \mathcal{K}_Θ), where

$$\mathcal{K}_\Xi = \{2^\omega \times \prod_{i \in \omega} C_i : C_i \text{ are compact subsets of } \alpha Z \text{ and almost all } C_i = \{\infty\}\};$$

$$\mathcal{K}_\Theta = \{2^\omega \times \prod_{i \in \omega} C_i : \exists i_0 \in \omega \text{ such that } C_{i_0} = \alpha Z, C_i = \{\infty\} \text{ for all } i > i_0 \text{ and } C_i \text{ is a compact subsets of } Z \text{ for every } i < i_0\}.$$

Lemma 6. *Suppose Z is a zero-dimensional locally metrizable locally compact s -pace. Then*

- (1) *the spaces $\Xi(Z)$ and $\Theta(Z)$ are topologically homogeneous;*
- (2) *$\Xi(Z)$ is a regular zero-dimensional k_ω -space while $\Theta(Z)$ is a totally disconnected k -space;*
- (3) *if Z is Lindelöf, then $\Xi(Z)$ and $\Theta(Z)$ are zero-dimensional \mathcal{MK}_ω -spaces with*

$$\chi(\Xi(Z)) = \chi(\Theta(Z)) \leq \mathfrak{d};$$
- (4) *$\Xi(Z)$ and $\Theta(Z)$ contain copies of the space αZ while $\Theta(Z)$ contains a closed copy of Z ;*
- (5) *$\text{cs}_\chi^*(\Xi(Z)) = \text{cs}_\chi^*(\Theta(Z)) = \text{cs}_\chi^*(\alpha Z)$, $\text{cs}_\chi(\Xi(Z)) = \text{cs}_\chi(\Theta(Z)) = \text{cs}_\chi(\alpha Z)$, $\text{sb}_\chi(\Theta(Z)) = \text{sb}_\chi(\alpha Z)$, and $\psi(\Xi(Z)) = \psi(\Theta(Z)) = \psi(\alpha Z)$;*
- (6) *the spaces $\Xi(Z)$ and $\Theta(Z)$ are sequential if and only if αZ is sequential;*
- (7) *if Z is not countably compact, then $\Xi(Z)$ contains a closed copies of S_2 and S_ω and $\Theta(Z)$ contains a closed copy of S_2 .*

Proof. (1) First we show that the space $\Xi(Z)$ is topologically homogeneous.

Given two points $(c, (z_i)_{i \in \omega}), (c', (z'_i)_{i \in \omega})$ of $\Xi(Z)$ we have to find a homeomorphism h of $\Xi(Z)$ with $h(c, (z_i)_{i \in \omega}) = (c', (z'_i)_{i \in \omega})$. Since the Cantor cube 2^ω is topologically homogeneous, we can assume that $c \neq c'$. Fix any disjoint closed-and-open neighborhoods U, U' of the points c, c' in 2^ω , respectively.

Consider the finite sets $I = \{i \in \omega : z_i \neq \infty\}$ and $I' = \{i \in \omega : z'_i \neq \infty\}$. Using the zero-dimensionality and the local metrizability of Z , for each $i \in I$ (resp. $i \in I'$) fix an open compact metrizable neighborhood U_i (resp. U'_i) of the point z_i (resp. z'_i) in Z . By the classical Brouwer Theorem [Ke, 7.4], the products $U \times \prod_{i \in I} U_i$ and $U' \times \prod_{i \in I'} U'_i$, being zero-dimensional compact metrizable spaces without isolated points, are homeomorphic to the Cantor cube 2^ω . Now the topological homogeneity of the Cantor cube implies the existence of a homeomorphism $f : U \times \prod_{i \in I} U_i \rightarrow U' \times \prod_{i \in I'} U'_i$ such that $f(c, (z_i)_{i \in I}) = (c', (z'_i)_{i \in I'})$. Let

$$W = \{(x, (x_i)_{i \in \omega}) \in \Xi(Z) : x \in U, x_i \in U_i \text{ for all } i \in I\} \text{ and}$$

$$W' = \{(x', (x'_i)_{i \in \omega}) \in \Xi(Z) : x' \in U', x'_i \in U'_i \text{ for all } i \in I'\}.$$

It follows that W, W' are disjoint open-and-closed subsets of $\Xi(Z)$. Let $\chi : \omega \setminus I' \rightarrow \omega \setminus I$ be a unique monotone bijection.

Now consider the homeomorphism $\tilde{f} : W \rightarrow W'$ assigning to a sequence $(x, (x_i)_{i \in \omega}) \in W$ the sequence $(x', (x'_i)_{i \in \omega}) \in W'$ where $(x', (x'_i)_{i \in I'}) = f(x, (x_i)_{i \in I})$ and $x'_i =$

$x_{\chi(i)}$ for $i \notin I'$. Finally, define a homeomorphism h of $\Xi(Z)$ letting

$$h(x) = \begin{cases} x & \text{if } x \notin W \cup W'; \\ \tilde{f}(x) & \text{if } x \in W; \\ \tilde{f}^{-1}(x) & \text{if } x \in W' \end{cases}$$

and observe that $h(c, (z_i)_{i \in \omega}) = (c', (z'_i)_{i \in \omega})$ which proves the topological homogeneity of the space $\Xi(Z)$.

Replacing $\Xi(Z)$ by $\Theta(Z)$ in the above proof, we shall get a proof of the topological homogeneity of $\Theta(Z)$.

The items (2–4) follow easily from the definitions of the spaces $\Xi(Z)$ and $\Theta(Z)$, the zero-dimensionality of αZ , and known properties of k_ω -spaces, see [FST] (to find a closed copy of Z in $\Theta(Z)$ consider the closed embedding $e : Z \rightarrow \Theta(Z)$, $e : z \mapsto (z, z_0, z, \infty, \infty, \dots)$, where z_0 is any fixed point of Z).

To prove (5) apply Proposition 3(6,8,9,10). (To calculate the cs^* -, cs -, and sb -characters of $\Theta(Z)$, observe that almost all members of any sequence $(a_n) \subset \Theta(Z)$ convergent to a point $a = (c, (z_i)) \in \Theta(Z)$ lie in the compactum $2^\omega \times \prod_{i \in \omega} C_i$, where C_i is a clopen neighborhood of z_i if $z_i \neq \infty$, $C_i = \alpha Z$ if $i = \min\{j \in \omega : z_j = \infty\}$ and $C_i = \{\infty\}$ otherwise. By Proposition 3(6), the cs^* -, cs -, and sb -characters of this compactum are equal to the corresponding characters of αZ .)

(6) Since the spaces $\Xi(Z)$ and $\Theta(Z)$ contain a copy of αZ , the sequentiality of $\Xi(Z)$ or $\Theta(Z)$ implies the sequentiality of αZ . Now suppose conversely that the space αZ is sequential. Then each compactum $K \in \mathcal{K}_\Xi \cup \mathcal{K}_\Theta$ is sequential since a finite product of sequential compacta is sequential, see [En₁, 3.10.I(b)]. Now the spaces $\Xi(Z)$ and $\Theta(Z)$ are sequential because they carry the inductive topologies with respect to the covers \mathcal{K}_Ξ , \mathcal{K}_Θ by sequential compacta.

(7) If Z is not countably compact, then it contains a countable closed discrete subspace $S \subset Z$ which can be thought as a sequence convergent to ∞ in αZ . It is easy to see that $\Xi(S)$ (resp. $\Theta(S)$) is a closed subset of $\Xi(Z)$ (resp. $\Theta(Z)$). Now it is quite easy to find closed copies of S_2 and S_ω in $\Xi(S)$ and a closed copy of S_2 in $\Theta(S)$. \square

With Lemma 6 at our disposal, we are able to finish the proof of Theorem 5. To construct the examples satisfying the conditions of Theorem 5(2,3), assume $\mathfrak{b} = \mathfrak{c}$ and use Proposition 4 to find a locally compact locally countable space Z whose one-point compactification αZ is sequential and satisfies $\aleph_0 = sb_\chi(\alpha Z) < \psi(\alpha Z) = \mathfrak{c}$. Applying Lemma 6 to this space Z , we conclude that the topologically homogeneous k -spaces $X_2 = \Xi(Z)$ and $X_3 = \Theta(Z)$ give us required examples.

The example of a countable topologically homogeneous k_ω -space X_1 with $sb_\chi(X_1) < \chi(X_1)$ can be constructed by analogy with the space $\Theta(\mathbb{N})$ (with that difference that there is no necessity to involve the Cantor cube) and is known in topology as the Ankhangel'ski-Franklin space, see [AF]. We briefly remind its construction. Let $S_0 = \{0, \frac{1}{n} : n \in \mathbb{N}\}$ be a convergent sequence and consider the countable space $X_1 = \{(x_i)_{i \in \omega} \in S_0^\omega : \exists n \in \omega \text{ such that } x_i \neq 0 \text{ iff } i < n\}$ endowed with the strongest topology inducing the product topology on each compactum $\prod_{i \in \omega} C_i$ for which there is $n \in \omega$ such that $C_n = S_0$, $C_i = \{0\}$ if $i > n$, and $C_i = \{x_i\}$ for some $x_i \in S_0 \setminus \{0\}$ if $i < n$. By analogy with the proof of Lemma 6 it can be shown that X_1 is a topologically homogeneous k_ω -space with $\aleph_0 = sb_\chi(X_1) < \chi(X_1) = \mathfrak{d}$ and $so(X_1) = \omega$.

Proof of Proposition 5. The equivalences (1) \Leftrightarrow (2) \Leftrightarrow (3) were proved by Lin [Lin, 3.13] in terms of (universally) *csf*-countable spaces. To prove the other equivalences apply

Lemma 7. *A Hausdorff topological space X is an α_4 -space provided one of the following conditions is satisfied:*

- (1) X is a Fréchet-Urysohn α_7 -space;
- (2) X is a Fréchet-Urysohn countably compact space;
- (3) $\text{sb}_\chi(X) < \mathfrak{p}$;
- (4) $\text{sb}_\chi(X) < \mathfrak{d}$, each point of X is regular G_δ , and X is c -sequential.

Proof. Fix any point $x \in X$ and a countable family $\{S_n\}_{n \in \omega}$ of sequences convergent to x in X . We have to find a sequence $S \subset X \setminus \{x\}$ convergent to x and meeting infinitely many sequences S_n . Using the countability of the set $\bigcup_{n \in \omega} S_n$ find a decreasing sequence $(U_n)_{n \in \omega}$ of closed neighborhoods of x in X such that $(\bigcap_{n \in \omega} U_n) \cap (\bigcup_{n \in \omega} S_n) = \{x\}$. Replacing each sequence S_n by its subsequence $S_n \cap U_n$, if necessary, we can assume that $S_n \subset U_n$.

(1) Assume that X is a Fréchet-Urysohn α_7 -space. Let $A = \{a \in X : a \text{ is the limit of a convergent sequence } S \subset X \text{ meeting infinitely many sequences } S_n\}$. It follows from our assumption on (S_n) and (U_n) that $A \subset \bigcap_{n \in \omega} U_n$.

It suffices to consider the non-trivial case when $x \notin A$. In this case x is a cluster point of A (otherwise X would be not α_7). Since X is Fréchet-Urysohn, there is a sequence $(a_n) \subset A$ convergent to x . By the definition of A , for every $n \in \omega$ there is a sequence $T_n \subset X$ convergent to a and meeting infinitely many sequences S_n . Without loss of generality, we can assume that $T_n \subset \bigcup_{i > n} S_i$ (because $a \in A \setminus \{x\}$ and thus $a \notin \bigcup_{n \in \omega} S_n$). It is easy to see that x is a cluster point of the set $\bigcup_{n \in \omega} T_n$. Since X is Fréchet-Urysohn, there is a sequence $T \subset \bigcup_{n \in \omega} T_n$ convergent to x .

Now it rests to show that the set T meets infinitely many sequences S_n . Assuming the converse we would find $n \in \omega$ such that $T \subset \bigcup_{i \leq n} S_i$. Then $T \subset \bigcup_{i \leq n} T_i$ which is not possible since $\bigcup_{i \leq n} T_i$ is a compact set failing to contain the point x .

(2) If X is Fréchet-Urysohn and countably compact, then it is sequentially compact and hence α_7 , which allows us to apply the previous item.

(3) Assume that $\text{sb}_\chi(X) < \mathfrak{p}$ and let \mathcal{N} be a sb -network at x of size $|\mathcal{N}| < \mathfrak{p}$. Without loss of generality, we can assume that the family \mathcal{N} is closed under finite intersections. Let $S = \bigcup_{n \in \omega} S_n$ and $F_{N,n} = N \cap (\bigcup_{i > n} S_i)$ for $N \in \mathcal{N}$ and $n \in \omega$. It is easy to see that the family $\mathcal{F} = \{F_{N,n} : N \in \mathcal{N}, n \in \omega\}$ consists of infinite subsets of S , has size $|\mathcal{F}| < \mathfrak{p}$, and is closed under finite intersection. Now the definition of the small cardinal \mathfrak{p} implies that this family \mathcal{F} has an infinite pseudo-intersection $T \subset S$. Then T is a sequence convergent to x and intersecting infinitely many sequences S_n . This shows that X is an α_4 -space.

(4) Assume that the space X is c -sequential, each point of X is regular G_δ , and $\text{sb}_\chi(X) < \mathfrak{d}$. In this case we can choose the sequence (U_n) to satisfy $\bigcap_{n \in \omega} U_n = \{x\}$. Fix an sb -network \mathcal{N} at x with $|\mathcal{N}| < \mathfrak{d}$. For every $n \in \omega$ write $S_n = \{x_{n,i} : i \in \mathbb{N}\}$. For each sequential barrier $N \in \mathcal{N}$ find a function $f_N : \omega \rightarrow \mathbb{N}$ such that $x_{n,i} \in N$ for every $n \in \omega$ and $i \geq f_N(n)$. The family of functions $\{f_N : N \in \mathcal{N}\}$ has size $< \mathfrak{d}$ and hence is not cofinal in \mathbb{N}^ω . Consequently, there is a function $f : \omega \rightarrow \mathbb{N}$ such that $f \not\leq f_N$ for each $N \in \mathcal{N}$. Now consider the sequence $S = \{x_{n,f(n)} : n \in \omega\}$. We claim that x is a cluster point of S . Indeed, given any neighborhood U of x ,

find a sequential barrier $N \in \mathcal{N}$ with $N \subset U$. Since $f \not\leq f_N$, there is $n \in \omega$ with $f(n) > f_N(n)$. It follows from the choice of the function f_N that $x_{n,f(n)} \in N \subset U$.

Since $S \setminus U_n$ is finite for every n , $\{x\} = \bigcap_{n \in \omega} U_n$ is a unique cluster point of S and thus $\{x\} \cup S$ is a closed subset of X . Now the c -sequentiality of X implies the existence of a sequence $T \subset S$ convergent to x . Since T meets infinitely many sequences S_n , the space X is α_4 . \square

Proof of Proposition 6. Suppose a space X has countable cs^* -character. The implications (1) \Rightarrow (2, 3, 4, 5) are trivial. The equivalence (1) \Leftrightarrow (2) follows from Proposition 1(2). To show that (3) \Rightarrow (2), apply Lemma 7 and Proposition 5(3 \Rightarrow 1).

To prove that (4) \Rightarrow (2) it suffices to apply Proposition 5(4 \Rightarrow 1) and observe that X is Fréchet-Urysohn provided $\chi(X) < \mathfrak{p}$ and X has countable tightness. This can be seen as follows.

Given a subset $A \subset X$ and a point $a \in \bar{A}$ from its closure, use the countable tightness of X to find a countable subset $N \subset A$ with $a \in \bar{N}$. Fix any neighborhood base \mathcal{B} at x of size $|\mathcal{B}| < \mathfrak{p}$. We can assume that \mathcal{B} is closed under finite intersections. By the definition of the small cardinal \mathfrak{p} , the family $\{B \cap N : B \in \mathcal{B}\}$ has infinite pseudo-intersection $S \subset N$. It is clear that $S \subset A$ is a sequence convergent to x , which proves that X is Fréchet-Urysohn.

(5) \Rightarrow (2). Assume that X is a sequential space containing no closed copies of S_ω and S_2 and such that each point of X is regular G_δ . Since X is sequential and contains no closed copy of S_2 , we may apply Lemma 2.5 [Lin] to conclude that X is Fréchet-Urysohn. Next, Theorem 3.6 of [Lin] implies that X is an α_4 -space. Finally apply Proposition 5 to conclude that X has countable sb-character and, being Fréchet-Urysohn, is first countable.

The final implication (6) \Rightarrow (2) follows from (5) \Rightarrow (2) and the well-known equality $\chi(S_\omega) = \chi(S_2) = \mathfrak{d}$.

Proof of Proposition 7. The first item of this proposition follows from Proposition 6(3 \Rightarrow 1) and the observation that each Fréchet-Urysohn countable compact space, being sequentially compact, is α_7 .

Now suppose that X is a dyadic compact with $cs_\chi^*(X) \leq \aleph_0$. If X is not metrizable, then it contains a copy of the one-point compactification αD of an uncountable discrete space D , see [En₁, 3.12.12(i)]. Then $cs_\chi^*(\alpha D) \leq cs_\chi^*(X) \leq \aleph_0$ and by the previous item, the space αD , being Fréchet-Urysohn and compact, is first-countable, which is a contradiction.

Proof of Proposition 8. Let D be a discrete space.

(1) Let $\kappa = cs_\chi^*(\alpha D)$ and λ_1 (λ_2) is the smallest weight of a (regular zero-dimensional) space X of size $|X| = |D|$, containing no non-trivial convergent sequence. To prove the first item of proposition 8 it suffices to verify that $\lambda_2 \leq \kappa \leq \lambda_1$. To show that $\lambda_2 \leq \kappa$, fix any cs^* -network \mathcal{N} at the unique non-isolated point ∞ of αD of size $|\mathcal{N}| \leq \kappa$. The algebra \mathcal{A} of subsets of D generated by the family $\{D \setminus N : N \in \mathcal{N}\}$ is a base of some zero-dimensional topology τ on D with $w(D, \tau) \leq \kappa$. We claim that the space D endowed with this topology contains no infinite convergent sequences. To get a contradiction, suppose that $S \subset D$ is an infinite sequence convergent to a point $a \in D \setminus S$. Then S converges to ∞ in αD and hence, there is an element $N \in \mathcal{N}$ such that $N \subset \alpha D \setminus \{a\}$ and $N \cap S$ is infinite. Consequently, $U = D \setminus N$ is a neighborhood of a in the topology τ such that $S \setminus U$

is infinite which contradicts to the fact that S converges to a . Now consider the equivalence relation \sim on D : $x \sim y$ provided for every $U \in \tau$ ($x \in U \Leftrightarrow (y \in U)$). Since the space (D, τ) has no infinite convergent sequences, each equivalence class $[x]_{\sim} \subset D$ is finite (because it carries the anti-discrete topology). Consequently, we can find a subset $X \subset D$ of size $|X| = |D|$ such that $x \not\sim y$ for any distinct points $x, y \in X$. Clearly that τ induces a zero-dimensional topology on X . It rests to verify that this topology is T_1 . Given any two distinct point $x, y \in X$ use $x \not\sim y$ to find an open set $U \in \mathcal{A}$ such that either $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$. Since $D \setminus U \in \mathcal{A}$, in both cases we find an open set $W \in \mathcal{A}$ such that $x \in W$ but $y \notin W$. It follows that X is a T_1 -space containing no non-trivial convergent sequence and thus $\lambda_2 \leq w(X) \leq |\mathcal{A}| \leq |N| \leq \kappa$.

To show that $\kappa \leq \lambda_1$, fix any topology τ on D such that $w(D, \tau) \leq \lambda_1$ and the space (D, τ) contains no non-trivial convergent sequences. Let \mathcal{B} be a base of the topology τ with $|\mathcal{B}| \leq \lambda_1$, closed under finite unions. We claim that the collection $\mathcal{N} = \{\alpha D \setminus B : B \in \mathcal{B}\}$ is a cs^* -network for αD at ∞ . Fix any neighborhood $U \subset \alpha D$ of ∞ and any sequence $S \subset D$ convergent to ∞ . Write $\{x_1, \dots, x_n\} = \alpha D \setminus U$ and by finite induction, for every $i \leq n$ find a neighborhood $B_i \in \mathcal{B}$ of x_i such that $S \setminus \bigcup_{j \leq i} B_j$ is infinite. Since \mathcal{B} is closed under finite unions, the set $N = \alpha D \setminus (B_1 \cup \dots \cup B_n)$ belongs to the family \mathcal{N} and has the properties: $N \subset U$ and $N \cap S$ is infinite, i.e., \mathcal{N} is a cs^* -network at ∞ in αD . Thus $\kappa \leq |\mathcal{N}| \leq |\mathcal{B}| \leq \lambda_1$. This finishes the proof of (1).

An obvious modification of the above argument gives also a proof of the item (2).

Proof of Proposition 9. Let D be an uncountable discrete space.

(1) The inequalities $\aleph_1 \cdot \log |D| \leq \text{cs}_\chi^*(\alpha D) \leq \text{cs}_\chi(\alpha D)$ follows from Propositions 7(1) and 1(2,4) yielding $|D| = \chi(\alpha D) = \text{sb}_\chi(\alpha D) \leq 2^{\text{cs}_\chi^*(\alpha D)}$. The inequality $\text{cs}_\chi(\alpha D) \leq \mathfrak{c} \cdot \text{cof}([\log |D|]^{\leq \omega})$ follows from proposition 8(2) and the observation that the product $\{0, 1\}^{\log |D|}$ endowed with the \aleph_0 -box product topology has weight $\leq \mathfrak{c} \cdot \text{cof}([\log |D|]^{\leq \omega})$. Under the \aleph_0 -box product topology on $\{0, 1\}^\kappa$ we understand the topology generated by the base consisting of the sets $\{f \in \{0, 1\}^\kappa : f|_C = g|_C\}$ where $g \in \{0, 1\}^\omega$ and C is a countable subset of κ .

The item (2) follows from (1) and the equality $\aleph_1 \cdot \log \kappa = 2^{\aleph_0} \cdot \min\{\kappa, (\log \kappa)^\omega\}$ holding under GCH for any infinite cardinal κ , see [HJ, 9.3.8]

REFERENCES

- [Ar₁] A.V. Arkhangelski, *Maps and spaces*, Uspekhi Mat. Nauk **21**:4 (1966), 133–184 (in Russian).
- [Ar₂] A.V. Arkhangelski, *The frequency spectrum of a topological space and the classification of spaces*, Soviet Math. Dokl. **13** (1972), 265–268 (in Russian).
- [Ar₃] A.V. Arkhangelski, *Structure and classification of topological spaces and cardinal invariants*, Russian Math. Surveys **33** (1978), 33–96 (in Russian).
- [Ar₄] A.V. Arkhangelski, *The frequency spectrum of a topological space and the product operation*, Trans. Moscow Math. Soc. **2** (1981), 163–200.
- [AF] A.V. Arkhangelski, S.P. Franklin, *Ordinal invariants for topological spaces*, Michigan Math. J. **15** (1968), 313–320; 506.
- [Ba₁] T. Banach, *On topological groups containing a Fréchet-Urysohn fan*, Matem. Studii **9**:2 (1998), 149–154.
- [Ba₂] T. Banach, *Topological classification of strong duals to nuclear (LF)-spaces*, Studia Math. **138**:3 (2000), 201–208.
- [Ba₃] T. Banach, *Topological classification of zero-dimensional \mathcal{M}_ω -groups*, Matem. Studii. **15**:1 (2001), 109–112.
- [Co] W. Comfort, *Topological groups*, in: K.Kunen, J.E.Vaughan (eds.) *Handbook of Set-theoretic Topology*, (Elsevier Science Publishers B.V., 1984), 1143–1263.
- [vD] E.K. van Douwen, *The integers and Topology*, in: K.Kunen, J.E.Vaughan (eds.), *Handbook of Set-Theoretic Topology* (North-Holland, Amsterdam, 1984), 111–167.
- [En₁] R. Engelking, *General Topology* (Mir, Moskow, 1986), (in Russian).
- [En₂] R. Engelking, *Theory of Dimensions, Finite and Infinite* (Heldermann Verlag, 1995).
- [FST] S.P. Franklin, B.V. Smith Thomas, *A survey of k_ω -spaces*, Topology Proc. **2** (1977), 111–124.
- [HJ] K. Hrbacek, T. Jech. *Introduction to Set Theory*, (Marcel Dekker, 1999).
- [Lin] S. Lin, *A note on Arens space and sequential fan*, Topology Appl. **81** (1997), 185–196.
- [Ke] A. Kechris, *Classical Descriptive Set Theory* (Springer, 1995).
- [MS] D.A. Martin, R.M. Solovay. *Internal Cohen Extensions*, Ann. Math. Logic. **2** (1970), 143–178.
- [Na] J. Nagata. *Generalized Metric Spaces, I*, in: K.Morita, J.Nagata (eds.), *Topics in General Topology* (Elsevier Sci. Publ., 1989), 315–366.
- [Ne] S. Nedev. *0-metrizable spaces*, Trudy Moskov. Mat. Obšč. **24** (1971) 201–236 (in Russian).
- [Ny₁] P. Nyikos. *Metrizability and the Fréchet-Urysohn property in topological groups*, Proc. Amer. Math. Soc. **83**:4 (1981), 793–801.
- [Ny₂] P.J. Nyikos. *Classes of compact sequential spaces*. In: J. Steprans, S. Watson (eds.), *Set Theory and its Applications*, Lect. Notes in Math. **1401** (Springer-Verlag, Berlin etc., 1989), 135–159.
- [PZ] I. Protasov, E. Zelenyuk, *Topologies on groups determined by sequences*, Matem. Studii, Monograph Series, **4** (VNTL, Lviv, 1999).
- [Sa] K. Sakai. *On \mathbb{R}^∞ -manifolds and Q^∞ -manifolds*, Topology Appl. **18** (1984), 69–79.
- [Sch] H. Schaefer, *Topological Vector Spaces* (The Macmillan Company, NY, 1966).
- [Shi] A. Shibakov, *Metrizability of sequential topological groups with point-countable k -network*, Proc. Amer. Math. Soc. **126** (1998), 943–947.
- [Tk] M. Tkachenko, *Introduction to topological groups*, Topology Appl. **86** (1998), 179–231.
- [Va] J.E. Vaughan. *Small uncountable cardinals and topology*, in: J. van Mill and G.M.Reed (eds.) *Open Problems in Topology* (North-Holland, Amsterdam, 1990), 195–216.
- [Ya] N.N. Yakovlev. *On the theory of g -metrizable spaces*, Soviet Math. Dokl. **17** (1976), 1217–1219. Russian original: Dokl. Akad. Nauk SSSR **229** (1976), 1330–1331.
- [Zd] L. Zdomsky, *Interplay of topological and algebraical structures on spaces with countable cs^* -character*, (in preparation).
- [Ze₁] E. Zelenyuk, *Topologies on groups determined by compact subspaces*, Matem. Studii **5** (1995) 5–16 (in Russian).
- [Ze₂] E. Zelenyuk, *On group operations on homogeneous spaces*, Proc. Amer. Math. Soc. (to appear).

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