

# ZEROS OF QUADRATIC FUNCTIONALS ON NON-SEPARABLE SPACES.

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## Abstract

We construct non-separable subspaces in the kernel of quadratic functional in some classes of complex and real Banach spaces.

## 1 Introduction

Of course, investigations of quadratic functionals is an old story [12], [6], [7]. According to [11], for any polynomial functional  $p$  with  $p(0) = 0$ , defined on an infinite dimensional complex linear space  $X$  there is an infinite dimensional subspace  $X_0$  in the kernel  $\ker(p) = p^{-1}(0)$  of  $p$ . Quantitative finite dimensional versions of this fact (estimations of  $\dim X_0$  depending on  $\dim X$  and the degree of polynomial) are contained in [1], [4], [5], [14].

The paper [2] started the consideration of subspaces in kernels of polynomials on non-separable spaces. In particular, the authors of [2] have proved that if a real Banach space  $X$  admits no positive quadratic continuous functional, then every quadratic continuous functional on  $X$  vanishes on some infinite dimensional subspace. They pose a problem whether in this statement one can replace “infinite dimensional” by “non-separable”? (See also [1], Question 4.8). Our note continues the investigations of [2]. In particular, we shall construct a non-separable subspace in the kernel of quadratic functional on a complex Banach space having weakly\* non-separable dual and on a real Banach space which has controlled separable projection property and admits no positive quadratic continuous functional. On the other hand, we construct a CH-example of a quadratic functional on the normed space  $l_1^f(\omega_1)$  whose kernel contains no nonseparable linear subspace.

We use the standard notation; in particular  $\text{dens}X$  stands for the density of a Banach space  $X$ ,  $F^\perp = \{x \in X : \forall f \in F f(x) = 0\}$  is the annihilator of a subspace  $F \subset X^*$  in  $X$ ,  $S(X)$  is the unit sphere of  $X$ , and  $[M]$  denotes the closed linear span of a subset  $M \subset X$ . We shall identify cardinals with initial ordinals and will denote by  $\bar{\alpha}$  the cardinality of an ordinal  $\alpha$ . Elements  $x_\alpha \in X$  form a *transfinite basic sequence* if there is a constant  $c > 0$  such that  $\|\sum_1^m a_i x_{\alpha_i}\| \leq c \|\sum_1^n a_i x_{\alpha_i}\|$  for any  $\alpha_1 < \alpha_2 < \dots < \alpha_m < \dots < \alpha_n$  and any numbers  $(a_i)$ . A *homogeneous quadratic functional* mean the functional  $q(x) = B(x, x)$ , where  $B(x, y)$  is a symmetric bilinear form defined on a linear space  $X$ .

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## 2 Complex case

In this section we consider Banach spaces with following property:

$$\text{dens}X/F^\perp \leq \text{card}F \quad \text{for any infinite subset } F \subset X^*. \quad (1)$$

In particular, all WCG spaces have the property (1) (see f.e. [10]).

**Proposition 1.** *Let  $q$  be a continuous homogeneous quadratic functional defined on a non-separable (real or complex) Banach space  $X$  with property (1). Then there exists a transfinite basic sequence  $x_\alpha \in S(X)$ ,  $\omega_0 \leq \alpha < \text{dens}X$ , such that for every finite collection of scalars  $(a_\alpha)$*

$$q\left(\sum a_\alpha x_\alpha\right) = \sum a_\alpha^2 q(x_\alpha). \quad (2)$$

*Proof.* We will provide the construction of  $x_\alpha$  by transfinite induction. Take an arbitrary element  $x_{\omega_0} \in S(X)$ .

If elements  $x_\alpha : \alpha < \beta$  are constructed, choose in the sphere  $S[x_\alpha : \alpha < \beta]$  a dense subset  $Y_\beta$  with  $\text{card}Y_\beta = \overline{\beta}$ . Let  $B(x, y)$  be the symmetric bilinear form, corresponding to functional  $q$ . Take an element  $x_\beta \in S(X)$  so that

$$B(x_\alpha, x_\beta) = 0 \quad \text{for all } \alpha < \beta \quad (3)$$

and

$$f_y(x_\beta) = 0 \quad \text{for all } y \in Y_\beta, \quad (4)$$

where  $f_y$  is a functional, attaining its norm on  $y$ .

Since  $X$  has property (1), this process can be continued up to  $\text{dens}X$ . The condition (2) follows from (3). The condition (4) guarantees that  $(x_\alpha)$  is a transfinite basic sequence.  $\square$

Note, that results, similar to Proposition 1, for usual sequences but for functionals of arbitrary degree was obtained in [11], [9]. Unfortunately, the methods of [11], [9] do not works for transfinite sequences. It easy to modify the proof of Proposition 1 for usual sequences.

**Proposition 2.** *The kernel  $\ker(q)$  of a continuous homogeneous quadratic functional  $q$ , defined on an infinite dimensional complex Banach space  $X$  with property (1), contains a subspace  $X_0 \subset \ker(q)$  with  $\text{dens}X_0 = \text{dens}X$ .*

*Proof.* Let  $(x_\alpha)$  be a transfinite sequence from Proposition 1. We can find a subset  $I \subset \{\alpha : \alpha < \text{dens}X\}$  of size  $|I| = \text{dens}X$  such that either  $q(x_\alpha) = 0$  for all  $\alpha \in I$  or else  $q(x_\alpha) \neq 0$  for all  $\alpha \in I$ .

In the first case put  $X_0 = [x_\alpha : \alpha \in I]$ . By (2),  $X_0 \subset \ker(q)$ . Since  $(x_\alpha)$  is a transfinite basic sequence,  $\text{dens}X_0 = \text{dens}X$ .

In the second case put  $z_\alpha = x_\alpha/q(x_\alpha)$  for  $\alpha \in I$ . Let  $X_0$  be the closed linear span of elements

$$z_{\omega_0} + iz_{\omega_0+1}, z_{\omega_0+2} + iz_{\omega_0+3}, \dots, z_{2\beta} + iz_{2\beta+1}, \dots$$

(we suppose the limit ordinals to be even). The condition (2) guarantees that  $X_0 \subset \ker(q)$ . Since  $(z_\beta)$  is a transfinite basic sequence,  $\text{dens}X_0 = \text{dens}X$ .  $\square$

The proofs of Propositions 1 and 2 imply

**Proposition 2'.** *If  $X^*$  is weakly\* non-separable, then the kernel of any complex homogeneous quadratic continuous functional on  $X$  contains a non-separable subspace.*

**Remark 1.** The condition  $\text{dens}X_0 = \text{dens}X$  in Proposition 2 cannot be improved to the separability of  $X/X_0$ . As a counterexample, is sufficiently to take  $X = l_2(\omega_1)$  and  $q(x) = \sum a_\alpha^2$ , where  $x = (a_\alpha : \alpha < \omega_1)$ . Every separable codimensional subspace  $X_0 \subset X$  contains uncountably many unit elements, so can not belong to  $\ker(q)$ .

Moreover, we shall show that the normed space  $l_1^f(\omega_1)$  of complex functions on  $(0, \omega_1)$  with finite support and endowed with the  $l_1$ -norm has the following surprising property.

**Proposition 3.** *Under Continuum Hypothesis there is a continuous quadratic functional  $q$  on  $l_1^f(\omega_1)$  whose kernel contains separable linear subspaces only.*

**Lemma 1.** *Suppose  $X$  is a complex normed space such that the kernel of each continuous quadratic functional  $q$  on  $X$  contains a linear non-separable subspace. Then for each bounded linear operator  $T : X \rightarrow l_2$  there is a non-separable subspace  $Y \subset X$  such that the closure  $H$  of  $T(Y)$  in  $l_2$  has infinite codimension in  $l_2$ .*

*Proof.* If the operator  $T$  has finite-dimensional range, then it has non-separable kernel  $Y$ . Consequently,  $Y$  is a non-separable subspace of  $X$  such that  $H$  has infinite codimension in  $l_2$ . So we can assume that  $T(X)$  is infinite-dimensional. In this case we can assume that  $T(X)$  is dense in  $l_2$ . On the space  $l_2$  consider the standard quadratic functional  $q(x) = \sum a_n^2$ , where  $x = (a_1, a_2, \dots)$ . It follows from our hypothesis that  $X$  contains a non-separable subspace  $Y \subset X$  lying in the kernel of the functional  $q \circ T$ . Then  $T(Y)$  lies in the kernel of  $q$ . We have to show that the closure  $H$  of  $T(Y)$  has infinite codimension in  $l_2$ . For this consider the real subspace  $\Re l_2 = \{x \in l_2 : x = \bar{x}\}$  of  $l_2$  and observe that  $H \cap \Re l_2 = \{0\}$ . This implies that  $H$  has infinite codimension in  $l_2$  as a real subspace and consequently,  $H$  is infinite codimensional in  $l_2$ .  $\square$

*Proof of Proposition 3.* Assume the Continuum Hypothesis. The family of closed subspaces of infinite codimension in the separable space  $l_2$  has size of continuum and thus can be enumerated as  $\{F_\alpha : \alpha < \omega_1\}$ . By transfinite induction we can choose a bounded transfinite sequence  $\{x_\alpha : \alpha < \omega_1\}$  in  $l_2$  such that  $x_\alpha \notin \cup_{\beta < \alpha} \text{lin}(F_\beta \cup \{x_\gamma : \gamma < \alpha\})$  for each ordinal  $\alpha < \omega_1$  (the existence of such a point  $x_\alpha$  follows from the Baire theorem since  $x_\alpha$  should avoid the countable union of linear spaces of infinite codimension in  $l_2$ ). Evidently, we can choose this sequence so that  $[x_\alpha : \alpha < \omega_1] = l_2$ . Now define a bounded operator  $T : l_1^f(\omega_1) \rightarrow l_2$  letting  $T(f) = \sum_{\alpha < \omega_1} f(\alpha)x_\alpha$  for a function  $f \in l_1^f(\omega_1)$ . Given a countable ordinal  $\alpha$  consider the characteristic function  $e_\alpha : \omega_1 \rightarrow \{0, 1\}$  of  $\{\alpha\}$  defined by  $\alpha^{-1}(1) = \{\alpha\}$ . This function  $e_\alpha$  is an element of  $l_1^f(\omega_1)$ . It follows from the choice of the sequence  $(x_\alpha)$  that  $T^{-1}(F_\alpha) \subset \text{lin}\{e_\beta : \beta \leq \alpha\}$  is separable in  $l_1^f(\omega_1)$ .

Assuming that the closure of  $T(Y)$  has infinite codimension in  $l_2$  for some non-separable subspace  $Y \subset l_1^f(\omega_1)$  find an ordinal  $\alpha < \omega_1$  with  $T(Y) \subset F_\alpha$  and observe that  $Y \subset T^{-1}(F_\alpha)$  is separable which is a contradiction.  $\square$

We do not know if Proposition 3 is true without the Continuum Hypothesis. Also we do not know if the normed space  $l_1^f(\omega_1)$  in this proposition can be replaced by the Banach space  $l_1(\omega_1)$ .

However the following fact is true.

**Proposition 4.** *Suppose that  $X$  is a Banach space whose all subspaces of infinite codimension are separable. Then there is a continuous quadratic polynomial  $q$  on  $X$  whose kernel  $q^{-1}(0)$  contains no non-separable linear subspace.*

*Proof.* Assuming the converse and applying Lemma 1 we conclude that for each bounded operator  $T : X \rightarrow l_2$  there is a non-separable subspace  $Y \subset X$  whose image  $T(Y)$  has infinitely codimensional closure in  $l_2$ .

Observe that the space  $X$  admits a countable family of linear functionals separating points of  $X$ . Indeed, take any countable linearly independent subset  $F$  in the unit sphere  $S^*$  of the dual space  $X^*$ . Then the subspace  $F^\perp = \{x \in X : \forall f \in F f(x) = 0\}$  of  $X$  has infinite codimension and thus is separable. Take any countable subset  $E \subset S^*$  separating points of  $F^\perp$ . Then the countable set  $F \cup E$  separates points of  $X$ . Using this countable set of functionals it is easy to construct an injective continuous operator  $T : X \rightarrow l_2$  (for example, put  $T(x) = (2^{-n}f_n(x))_{n \in \omega}$  where  $\{f_n : n \in \omega\}$  is any enumeration of  $F \cup E$ ).

It follows from the above discussion that  $X$  contains a non-separable subspace  $Y$  such that the closure of  $T(Y)$  has infinite codimension in  $l_2$ . Then  $Y$  has infinite codimension in  $X$  and hence must be separable. This is a contradiction.  $\square$

In light of the previous proposition it should be mentioned that *the existence of a non-separable Banach space without non-separable infinitely codimensional subspaces* is a well-known open problem.

Now we consider the zeros of functionals, generated by sequence of linear functionals.

**Proposition 5.** *Let  $X$  be a (real or complex) Banach space with condition (1) and  $\varphi(t_1, t_2, \dots)$  be arbitrary function of countable many variables such that  $\varphi(0, 0, \dots) = 0$ . Then for any sequence  $f_1, f_2, \dots$  from  $X^*$  the kernel of the functional  $\varphi(f_1(x), f_2(x), \dots)$  contains a separable codimensional subspace.*

*Proof.* Let  $X_0$  be the subspace of common zeros of all  $f_n$ . It is clear that  $X_0 \subset \ker \varphi(f_1(x), f_2(x), \dots)$ . Since  $X$  has property (1),  $X/X_0$  is separable.  $\square$

Given a (real or complex) Banach space  $X$  denote by  $\mathcal{P}_A(X)$  the space of approximable functionals equal to the completion of finite sums of finite products of linear functionals in the uniform topology [8], p.85.

**Corollary.** *If  $X$  is a (real or complex) Banach space  $X$  with condition (1), then the kernel of each functional from  $\mathcal{P}_A(X)$  contains a separable codimensional subspace.*

### 3 Real case

In this section we consider real Banach spaces.

Following [13] we say that a Banach space  $X$  has *controlled separable projection property* (CSPS), if for every countable subsets  $E \subset X$  and  $F \subset X^*$  there exists a separable valued projection  $P$  in  $X$  with  $\|P\| = 1$ ,  $PX \supset E$  and  $P^*X^* \supset F$ .

However, the condition  $\|P\| = 1$  is not essential (see [10]). Every WCG space has CSPP. This property is stronger than separable complementation property; as an example one can take  $l_1(\omega_1)$ . This space, as any space with unconditional basis, has the separable complementation property but, because  $l_1(\omega_1)^*$  is weakly\* separable, has no CSPP. We do not know about the connection between CSPS and property (1).

Let us make two simple obvious notes.

**Lemma 2.** *Let  $X = Y \oplus Z$ ,  $Y$  and  $Z$  admit positive quadratic continuous functionals. Then  $X$  admits a positive quadratic continuous functional as well. In particular, if  $Y$  is separable and  $Z$  admit a positive quadratic continuous functional, then  $X$  admits a positive quadratic continuous functional as well.*

**Lemma 3.** *If a real valued continuous function  $f(x)$  takes on a two-dimensional normed space values of distinct signs, then  $f$  vanishes on some nonzero element.*

**Proposition 6.** *Let  $X$  be a real Banach space with CSPP. If  $X$  admits no positive quadratic continuous functional, then every quadratic continuous functional  $q$  on  $X$  vanishes on some non-separable subspace.*

*Proof.* Let  $B(x, y)$  be the symmetric bilinear form corresponding to  $q$ . Let us construct, by induction, a transfinite basic sequence of elements  $x_\alpha : 1 \leq \alpha < \omega_1$  in  $X$  such that for all  $\alpha \geq \beta$

$$B(x_\alpha, x_\beta) = 0. \quad (5)$$

By Lemma 3, there exists an element  $x_1 \neq 0$  for which  $B(x_1, x_1) = 0$ .

Suppose elements  $x_\beta : 1 \leq \beta < \alpha$ ,  $\alpha < \omega_1$ , are constructed. Putting  $E = \{x_\beta : \beta < \alpha\}$  and  $F = \{f_\beta : \beta < \alpha\}$ , where  $f_\beta(x) = B(x_\beta, x)$ , we find a separable valued projection  $P$  in  $X$  with  $\|P\| = 1$ ,  $PX \supset E$  and  $P^*X^* \supset F$ . Since  $X$  admits no positive quadratic continuous functional, by Lemmas 2 and 3, there is an element  $x_\alpha \in \ker P$ ,  $x_\alpha \neq 0$  for which  $B(x_\beta, x_\alpha) = 0$  for  $\beta < \alpha$ . Obviously, the condition (5) for  $x_\beta$  is satisfied. By construction,  $(x_\alpha)$  form transfinite basic sequence, hence  $X_0 = [x_\alpha : \alpha < \omega_1]$  is non separable. Condition (5) guarantees that  $X_0 \subset \ker(q)$ .  $\square$

**Remark 3.** One cannot improve in Proposition 6  $\omega_1$  to a larger cardinal. As a counterexample we can take  $l_3(\omega_1) \oplus l_2(\omega_2)$ . Proposition 6 is connected with the following three space problem. *Assume that for a subspace  $Y$  of a Banach space  $X$  there exist continuous linear injective operators from  $Y$  and  $X/Y$  into a Hilbert space. Does there exist a continuous linear injective operator from  $X$  into a Hilbert space?* In particular, let  $X/Y$  has weakly\* separable dual and there is a continuous linear injective operator from  $Y$  into a Hilbert space. Does there exist a continuous linear injective operator from  $X$  into a Hilbert space?

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