FELL TOPOLOGY ON HYPERSPACES OF LOCALLY COMPACT SPACES

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Abstract. We study general-topological and infinite-dimensional properties of the Fell topology on the hyperspace $\mathrm{Cld}^*_F(X)$ of closed subsets of a locally compact paracompact space X.

In this paper we shall study the Fell topology of the hyperspace $\mathrm{Cld}^*_F(X)$ of closed subsets of a regular locally compact non-compact space X. We recall that the Fell topology is generated by the subbase consisting of sets

$$\langle U\rangle^+ = \{F \in \mathrm{Cld}^*_F(X) : F \cap U \neq \emptyset\} \text{ and } \langle K\rangle^- = \{F \in \mathrm{Cld}^*_F(X) : F \cap K = \emptyset\}$$

where U and K run over open and compact subsets of X, respectively. If instead of compact sets, K runs over the family of all closed subsets of X, then we get the hyperspace $\mathrm{Cld}^*_V(X)$ of closed subsets endowed with the *Vietoris topology*. The study of the Vietoris topology of hyperspaces is a classical subject of general and categorial topology, see [IN]. The empty set is always an isolated point of $\mathrm{Cld}^*_V(X)$. Because of that studying the Vietoris topology, one usually considers the hyperspace $\mathrm{Cld}_V(X) = \mathrm{Cld}^*_V(X)\setminus\{\emptyset\}$ of non-empty closed subsets of X. In contrast, the empty set is not isolated in $\mathrm{Cld}^*_F(X)$ if X is not compact.

It is clear that the Fell topology is weaker than the Vietoris one. On the other hand, these two topologies coincide if X is compact. In case of locally compact non-compact space X the Fell topology on $\mathrm{Cld}^*(X)$ is a bit better than the Vietoris topology since the former topology always is compact [Fe] (cf. [Be]). Moreover, if X is metrizable and separable, then $\mathrm{Cld}^*_F(X)$ is compact metrizable while $\mathrm{Cld}^*_V(X)$ fails to be metrizable (unless X is compact).

Yet, there is a close interplay between these two topologies: taking the one-point compactification $\bar{X} = X \cup \{\infty\}$ of a non-compact locally compact space X and identifying each closed subset $F \in \mathrm{Cld}^*_F(X)$ with the compact subset $F \cup \{\infty\}$ of $\bar{X}$, we can identify the hyperspace $\mathrm{Cld}^*_F(X)$ with the closed subset $\{K \in \mathrm{Cld}_V(\alpha X) : \infty \in K\}$ of $\mathrm{Cld}_V(\alpha X)$. Since this set is a retract of $\mathrm{Cld}_V(\alpha X)$ the hyperspaces $\mathrm{Cld}^*_F(X)$ and $\mathrm{Cld}_V(\alpha X)$ have many common properties. Yet, the hyperspace $\mathrm{Cld}^*_F(X)$ differs substantially from $\mathrm{Cld}_V(\alpha X)$.

The difference appears already at the level of discrete spaces. Namely, assigning to each subset A of X its characteristic function $\chi_A : X \to \{0,1\}$ we can identify the hyperspace $\mathrm{Cld}^*_F(X)$ with the Cantor cube $\{0,1\}^X$. In contrast, for an uncountable X the hyperspace $\mathrm{Cld}^*_V(\alpha X)$ has uncountable Suslin number and thus is not even dyadic.

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Having in mind nice properties of the hyperspace $\mathrm{Cld}^*_F(X) \cong \{0,1\}^X$ for discrete X, I. Protasov asked if the hyperspace $\mathrm{Cld}^*_F(X)$ still is countably cellular for any sufficiently nice locally compact space X, for example, for any locally compact topological group. This question was motivated by an earlier Protasov's result [Pro_1] asserting that the subspace $\mathrm{Clgr}_F(X) \subset \mathrm{Cld}^*_F(G)$ of closed subgroups of an arbitrary compact group G has countable cellularity. It should be mentioned that for any infinite cardinal κ there is a discrete (non-commutative) group G whose hyperspace $\mathrm{Clgr}_V(X)$ has cellularity equal to κ [Pro_2].

We shall try to answer the Protasov question in two directions: the general-topological and infinite-dimensional.

1. General-topological properties of hyperspaces

In this section we shall calculate some basic cardinal characteristics of hyperspaces $\mathrm{Cld}^*_F(X)$ for paracompact locally compact spaces X. The paracompactness of X will be helpful because of the following well-known decomposition result, see [En, Theorem 5.1.27].

Proposition 1. *Each paracompact locally compact space X admits a disjoint cover by closed-and-open σ-compact subsets.*

This decomposition result allows us to detect a product structure of the hyperspace $\mathrm{Cld}^*_F(X)$ established in the following proposition whose proof is left to the reader.

Proposition 2. *Assume that a locally compact space X is the disjoint union $X = \sqcup_{i\in I} X_i$ of closed-and-open subsets X_i of X. Then the map*

$$h : \mathrm{Cld}^*_F(X) \to \prod_{i\in I} \mathrm{Cld}^*_F(X_i), \quad h : F \mapsto (F \cap X_i)_{i\in I},$$

is a homeomorphism.

Corollary 1. *If X is a metrizable locally compact space, then the hyperspace $\mathrm{Cld}^*_F(X)$ is homeomorphic to a product of metrizable compacta.*

Proof. Being paracompact and locally compact, the space X can be written as the disjoint sum $X = \sqcup_{i\in I} X_i$ of clopen σ-compact subsets X_i of X according to Proposition 1.

Assigning to each closed subset $F \subset X$ the family of intersections $h(F) = (F \cap X_i)_{i\in I} \in \prod_{i\in I} \mathrm{Cld}^*_F(X_i)$, we define a homeomorphism $h : \mathrm{Cld}^*_F(X) \to \prod_{i\in I} \mathrm{Cld}^*_F(X_i)$.

If X is metrizable, then each subspace X_i, $i \in I$, is metrizable. The σ-compactness of X_i implies the metrizability of the one-point compactification αX_i of X_i and the metrizability of the hyperspace $\mathrm{Cld}^*_V(\alpha X_i)$. Since $\mathrm{Cld}^*_F(X_i)$ is homeomorphic to a closed subspace of $\mathrm{Cld}^*_V(\alpha X_i)$, the latter space is metrizable and hence $\mathrm{Cld}^*_F(X)$ is homeomorphic to the product $\prod_{i\in I} \mathrm{Cld}^*_F(X_i)$ of metrizable compacta. □

Proposition 2 is very helpful in estimating cardinal characteristics of the hyperspaces $\mathrm{Cld}^*_F(X)$ of paracompact locally compact spaces.

We recall that for a topological space X

- $w(X)$, the *weight* of X, is the smallest size of a base of the topology of X;
- $d(X) = \min\{|Y| : \text{Y is dense in } X\}$ is the *density* of X ;

- $c(X)$, *the cellularity* or *Souslin number* of X, is the smallest cardinal κ such that X contains no collection $\mathcal{U}$ of pairwise disjoint nonempty open subsets of X with $|\mathcal{U}| > \kappa$;
- $sh(X)$, the *Shanin number* of X, is the smallest cardinal κ such that any collection $\mathcal{U}$ of nonempty open subsets of X with $|\mathcal{U}| > \tau$ contains a subcollection $\mathcal{V}$ of size $|\mathcal{V}| > \kappa$ such that $\cap\mathcal{V} \neq \emptyset$;
- $L(X)$, the *Lindelöf number* of X, is the smallest cardinal κ such that each open cover of X contains a subcover of X with size $\le \kappa$;
- $wL(X)$, the *weak Lindelöf number* of X, is the smallest cardinal κ such that each open cover of X contains a subcollection of size $\le \kappa$ with dense union in X.

It is clear that $wL(X) \le c(X) \le sh(X) \le d(X) \le w(X)$ for each topological space X. Because of the product structure of hyperspaces, it is important to discuss the productivity properties of cardinal invariants. The Shanin number is productive in the sense that $sh(\prod_{i\in I} X_i) = \sup_{i\in I} sh(X_i)$ for any family $(X_i)_{i\in I}$ of topological spaces, see [En, 2.7.11]. According to the famous Marczewski-Pondiczery Theorem [En, Theorem 2.3.15], the density has a somewhat weaker property: $d(\prod_{i\in I} X_i) \le \sup\{\log|I|, d(X_i) : i \in I\}$, where $\log(\kappa) = \min\{\lambda : \kappa \le 2^\lambda\}$. The productivity of the Suslin number is a subtle question. Under (MA+$\neg$CH) any compact space with countable Suslin number has countable Shanin number and thus an arbitrary product of countably cellular Tychonov spaces has countable Suslin number, see [Ar]. On the other hand, if there exists a Suslin line L (= a linearly ordered non-separable space with countable Suslin number), then $c(L \times L) > c(L) = \aleph_0$, see [En, 2.7.10 b]. Yet, for any cardinal κ the the product X^κ has the Suslin number $c(X^\kappa) = c(X^{<\omega})$, where $X^{<\omega} = \sqcup_{n\in\omega} X^n$ stands for the discrete topological sum of finite powers of X, see [En, 2.7.10 d].

It turns out that a number of cardinal topological invariants of the hyperspace $\mathrm{Cld}^*_F(X)$ depend on local versions of these cardinal invariants for X.

Given a topological cardinal invariant $f(\cdot)$ we define its local version $lf(\cdot)$ letting $lf(X)$ be the smallest cardinal κ such that each point $x \in X$ has a neighborhood $U \subset X$ with $f(U) \le \kappa$. In such a way we define the *local cellularity* $lc(X)$, *local density* $ld(X)$ and *local Shanin number* $lsh(X)$ of a topological space X. Cardinal functions and their local versions are connected via the weak Lindelöf number: $f(X) = lf(X) \cdot wL(X)$, where f is the cellularity, density or the Shanin number.

The following theorem expresses cardinal invariants of $\mathrm{Cld}^*_F(X)$ via their local versions.

Theorem 1. *Let X be a locally compact paracompact space X. Then*

(1) $sh(\mathrm{Cld}^*_F(X)) = lsh(X)$;
(2) $c(\mathrm{Cld}^*_F(X)) = lc(X^{<\omega})$;
(3) $d(\mathrm{Cld}^*_F(X)) \le ld(X) \cdot \log c(X)$.

For compact spaces the local and global cardinal invariants coincide so we get the following corollary concerning the Vietoris topology of hyperspaces $\mathrm{Cld}_V(X)$ of compact Hausdorff spaces X. Here $\mathrm{Cld}_V(X) = \mathrm{Cld}^*_V(X) \setminus \{\emptyset\}$ is the family of all non-empty closed subsets of X.

Corollary 2. *If X is a compact Hausdorff space, then*

(1) $sh(\mathrm{Cld}_V(X)) = \mathrm{lsh}(X) = sh(X)$;
(2) $c(\mathrm{Cld}_V(X)) = lc(X^{<\omega}) = c(X^{<\omega})$;

(3) $d(\mathrm{Cld}_V(X)) \le d(X)$.

The second item of this corollary was proved by V. Fedorchuk and S. Todorčevič [FT] while other items are rather trivial and follow from general results about normal topological functors, see [TZ].

Under (MA+¬CH) the first assertion of Theorem 1 can be a bit improved. Under this assumption each compact space with countable Souslin number has countable Shanin number, see [Ar]. Consequently, we get

Corollary 3. *Under (MA+¬CH) for any locally compact paracompact space X with countable local Suslin number $lc(X) \le \aleph_0$ the hyperspace $\mathrm{Cld}^*_F(X)$ has countable Shanin number $sh(\mathrm{Cld}^*_F(X)) \le \aleph_0$.*

Now we apply the obtained results to the hyperspaces of locally compact topological groups answering the question of I. Protasov.

Corollary 4. *Each locally compact group G is paracompact and has local Shanin number $lsh(G) \le \aleph_0$. Consequently, $sh(\mathrm{Cld}^*_F(G)) \le \aleph_0$.*

For the proof we need the following result about topological groups, see [He, §9].

Lemma 1. *Every locally compact group is homeomorphic to a product of the form $\mathbb{R}^n \times K \times D$, where $n \in \mathbb{N} \cup \{0\}$, K is a compact group, and D is a discrete space.*

Proof. The product of paracompact $\mathbb{R}^n$ and compact K is paracompact by [En, Theorem 5.1.36], and the product of paracompact and discrete space is also paracompact. Whence, we obtain that each locally compact group is paracompact. To prove that $lsh(G) \le \aleph_0$, observe that each point of G has a neighborhood U homeomorphic to $\mathbb{R}^n \times K$. Whence, $sh(U) = sh(\mathbb{R}^n \times K) \le \aleph_0$, since each compact group is dyadic by Ivanovskiĭ-Vilenkin-Kuzminov theorem [To, Section 18] and thus has countable Shanin number. □

2. Proof of Theorem 1

To prove Theorem 1 we need the following lemma.

Lemma 2. *If X is a locally compact space, then $d(\mathrm{Cld}^*_F(X)) \le d(X)$, $lsh(X) \le sh(\mathrm{Cld}^*_F(X)) \le sh(X)$, and $lc(X^{<\omega}) \le c(\mathrm{Cld}^*_F(X)) \le c(X^{<\omega})$.*

Proof. For each $n \in \mathbb{N}$ let $\mathrm{Fin}_n(X) = \{F \in \mathrm{Cld}^*_F(X) : |F| \le n\}$ and note $\mathrm{Fin}_n(X)$ is the image of the power X^n under the continuous map $(x_1, \dots, x_n) \mapsto \{x_1, \dots, x_n\}$. Also observe that the union $\mathrm{Fin}^*(X) = \bigcup_{n\in\omega} \mathrm{Fin}_n(X)$ is dense in $\mathrm{Cld}^*_F(X)$.

Now we see that

$$d(\mathrm{Cld}^*_F(X)) \le d(\mathrm{Fin}^*(X)) \le \sum_{n\in\omega} d(\mathrm{Fin}_n(X)) \le \sum_{n\in\omega} d(X^n) = d(X).$$

The same argument yields the inequalities $c(\mathrm{Cld}^*_F(X)) \le \sum_{n\in\mathbb{N}} c(X^n) = c(X^{<\omega})$ and $sh(\mathrm{Cld}^*_F(X)) \le sh(X)$.

Next, we prove that $lsh(X) \le sh(\mathrm{Cld}^*_F(X))$ for any locally compact space X. Let $\kappa = sh(\mathrm{Cld}^*_F(X))$. Fix any point $x \in X$ and take any open neighborhood $O(x) \subset X$ having compact closure K in X. To show that $sh(O(x)) \le \kappa$, take any family $\mathcal{U}$ of non-empty open subsets of $O(x)$ with $|\mathcal{U}| > \kappa$. For each $U = \mathcal{U}$ consider the open set $\tilde{U} = \{F \in \mathrm{Cld}^*_F(X) : F \cap U \ne \emptyset \text{ and } F \cap (K \setminus U) = \emptyset\}$ in $\mathrm{Cld}^*_F(X)$. Since $sh(\mathrm{Cld}^*_F(X)) = \kappa$, the family $\mathcal{U}$ contains a subfamily $\mathcal{V}$ of size $|\mathcal{V}| > \kappa$ such that the intersection $\bigcap_{V\in\mathcal{V}}\tilde{V}$ contains some non-empty element $F \in \mathrm{Cld}^*_F(X)$. Then $F\cap V\neq\emptyset$ and $F\cap K\subset V$ for each $V\in\mathcal{V}$, which implies $\emptyset\neq F\cap K\subset\cap\mathcal{V}$.

Finally, we show that $lc(X^{<\omega})\le c(\mathrm{Cld}^*_F(X))$. Let $\kappa=c(\mathrm{Cld}^*_F(X))$. The inequality $lc(X^{<\omega})\le\kappa$ will follow as soon as we show that $c(\overline{W}^n)\le\kappa$ for each $n\in\mathbb{N}$ and each open set $W\subset X$ with compact closure $K=\overline{W}$ in X. For this we first prove that $c(\mathrm{Cld}_V(K))\le\kappa$. We should show that any family $\mathfrak{U}$ of non-empty open subsets of $\mathrm{Cld}_V(K)$ with $|\mathfrak{U}|>\kappa$ contains two elements with non-empty intersection.

Replacing each element $\mathcal{U}\in\mathfrak{U}$ by a smaller subset, we can assume that $\mathcal{U}$ is of the basic form

$$\mathcal{U}=\langle U_1\rangle^+\cup\cdots\cup\langle U_n\rangle^+\cup\langle K\setminus\bigcup_{i=1}^n U_i\rangle^-$$

for some disjoint open sets $U_1,\dots,U_n\subset W$. Considering the latter basic set as a basic subset of the space $\mathrm{Cld}^*_F(X)$ we extend each $\mathcal{U}\in\mathfrak{U}$ to a basic open subset $\tilde{\mathcal{U}}$ of $\mathrm{Cld}^*_F(X)$. Taking into account that $c(\mathrm{Cld}^*_F(X))=\kappa<|\mathfrak{U}|$, we can find two elements $\mathcal{U},\mathcal{V}\in\mathfrak{U}$ such that $\tilde{\mathcal{U}}\cap\tilde{\mathcal{V}}$ contains some closed set $F\in\mathrm{Cld}^*_F(X)$. Then $F\cap K\in\mathcal{U}\cap\mathcal{V}$, which finishes the proof of the inequality $c(\mathrm{Cld}_V(K))\le\kappa$.

Now we can apply the equality $c(\mathrm{Cld}_V(K))=c(K^\omega)$ proved in [FT] to conclude that $c(\overline{W}^n)=c(K^n)\le c(K^\omega)=c(\mathrm{Cld}_V(X))\le\kappa$. □

Proof of Theorem 1. Let X be a paracompact locally compact space. According to Proposition 1, X can be presented as the disjoint union of a family $\{X_i\}_{i\in I}$ clopen σ-compact subsets X_i of X.

Because of the σ-compactness, each space X_i, $i\in I$, is Lindelöf and thus its cardinal characteristics (cellularity, Shanin number, and density) coincide with their local versions. More precisely, $d(X_i)=ld(X_i)$, $sh(X_i)=lsh(X_i)$, and $c(X_i^{<\omega})=lc(X_i^{<\omega})$. Applying Lemma 2, we get $d(\mathrm{Cld}^*_F(X_i))\le d(X_i)$, $sh(\mathrm{Cld}^*_F(X_i))=sh(X_i)$ and $c(\mathrm{Cld}^*_F(X_i))=c(X_i^{<\omega})$.

Assigning to each closed subset $F\subset X$ the family of intersections $h(F)=(F\cap X_i)_{i\in I}\in\prod_{i\in I}\mathrm{Cld}^*_F(X_i)$, we define a homeomorphism $h:\mathrm{Cld}^*_F(X)\to\prod_{i\in I}\mathrm{Cld}^*_F(X_i)$.

Now let us prove the clauses of Theorem 1.

1. We shall show that $sh(\mathrm{Cld}^*_F(X))=lsh(X)$. In fact, the inequality $lsh(X)\le sh(\mathrm{Cld}^*_F(X))$ was proved in Lemma 2. To prove the inverse inequality, let $\kappa=\mathrm{lsh}(X)$ and note that $sh(\mathrm{Cld}^*_F(X_i))\le sh(X_i)=\mathrm{lsh}(X_i)\le lsh(X)\le\kappa$. Now we can use the productivity of the Shanin number to conclude that

$$sh(\mathrm{Cld}^*_F(X))=sh(\prod_{i\in I}\mathrm{Cld}^*_F(X_i))\le\sup_{i\in I}sh(\mathrm{Cld}^*_F(X_i))\le\kappa=lsh(X).$$

2. Next, we show that $c(\mathrm{Cld}^*_F(X))\le lc(X^{<\omega})$ (the inverse inequality was proved in Lemma 2). Let $\kappa=lc(X^{<\omega})$. Since $\mathrm{Cld}^*_F(X)$ is homeomorphic to $\prod_{i\in I}\mathrm{Cld}^*_F(X_i)$, the inequality $c(\mathrm{Cld}^*_F(X))\le\kappa$ will follow as soon as we prove that $c(\prod_{i\in E}\mathrm{Cld}^*_F(X_i))\le\kappa$ for each finite index subset $E\subset I$. Let $X_E=\bigcup_{i\in F}X_i$ and observe that X_E is a locally compact σ-compact space and so is the space $X_E^{<\omega}$. Then $c(X_E^{<\omega})=lc(X_E^{<\omega})\le lc(X^{<\omega})=\kappa$ and thus $c(\mathrm{Cld}^*_F(X_E))\le\kappa$ by Lemma 2.

Observing that $\prod_{i\in E}\mathrm{Cld}^*_F(X_i)$ is homeomorphic to $\mathrm{Cld}^*_F(\bigcup_{i\in E}X_i)=\mathrm{Cld}^*_F(X)$, we get $c(\prod_{i\in E}\mathrm{Cld}^*_F(X_i))=c(\mathrm{Cld}^*_F(X_F))\le\kappa$.

3. Finally we show that $d(\mathrm{Cld}^*_F(X)) \le ld(X)\cdot \log wL(X)$. Let $\kappa = ld(X)\cdot \log wL(X)$. Observe that $\log|I| \le \log wL(X) \le \kappa$ and thus $|I| \le 2^\kappa$. For each $i\in I$ we get $d(\mathrm{Cld}^*_F(X_i)) \le d(X_i) = ld(X_i) \le ld(X) \le \kappa$. Now Marczewski-Pondiczery Theorem [En, Theorem 2.3.15] implies that $d(\mathrm{Cld}^*_F(X)) = d(\prod_{i\in I}\mathrm{Cld}^*_F(X_i)) \le \kappa$.

3. Infinite-dimensional properties of hyperspaces

In this section we shall show that the hyperspaces $\mathrm{Cld}^*_F(X)$ of metrizable locally compact spaces often are homeomorphic to some well-known model spaces of (infinite-dimensional) topology whose definitions we are going to recall now. Let

- $2 = \{0,1\}$ be the discrete two-point space;
- $I = [0,1]$ denote the closed interval;
- 2^ω be the Cantor cube;
- 2^κ be the Cantor cube of weight κ;
- $Q = I^\omega$ be the Hilbert cube;
- I^κ be the Tychonov cube;
- $s = (0,1)^\omega$ be the pseudointerior of the Hilbert cube;
- $\Sigma = \{(x_i)\in I^\omega : \sup\{x_i : i\in\omega\} < 1\}$ is the radial interior of I^ω;
- $\sigma = \{(x_i)\in I^\omega : x_i = 0$ for almost all $i\}$.

The space σ is a particular case of the following construction. Given a pointed space $(X,*)$ and two cardinals κ,λ let

$$X^\lambda_{<\kappa} = \{(x_\alpha)_{\alpha\in\lambda}\in X^\lambda : |\{\alpha\in\lambda : x_\alpha \ne *\}| < \kappa\}.$$

Here we identify cardinals with initial ordinals. In the sequel we consider the spaces $2=\{0,1\}$ and $I=[0,1]$ as pointed spaces with zero as a distinguished point. Thus $\sigma = I^\omega_{<\omega}$ while Σ is homeomorphic to $Q^\omega_{<\omega}$ as well as to $Q\setminus s$, the pseudo-boundary of the Hilbert cube.

The study of the structure of hyperspaces from the view-point of infinite-dimensional topology is rather a classical topic in topology. Together with the hyperspace $\mathrm{Cld}^*(X)$ some of its important subspaces were studied as well. We shall be interested in the subspaces

- $\mathrm{Cld}(X)$ of all non-empty closed subsets of X;
- $\mathrm{Comp}^*(X)$ of all compact subsets of X;
- $\mathrm{Comp}(X) = \mathrm{Comp}^*(X)\cap\mathrm{Cld}(X) = \mathrm{Comp}^*(X)\setminus\{\emptyset\}$;
- $\mathrm{Fin}^*(X)$ of all finite subsets of X;
- $\mathrm{Fin}(X) = \mathrm{Fin}^*(X)\cap\mathrm{Cld}(X) = \mathrm{Fin}^*(X)\setminus\{\emptyset\}$;
- $\mathrm{Cld}^*_{<\kappa}(X) = \{A\in\mathrm{Cld}^*_F(X)$: each open cover of A contains a subcover having size $<\kappa\}$, where κ is a cardinal.

It should be mentioned that $\mathrm{Comp}^*(X) = \mathrm{Cld}^*_{<\omega}(X)$ and $\mathrm{Fin}^*(X) = \bigcup_{n\in\omega}\mathrm{Cld}^*_{<n}(X)$.

Considering these spaces as subspaces of $\mathrm{Cld}^*_V(X)$ or $\mathrm{Cld}^*_F(X)$ we shall endow them with a subscript V or F indicating the choice of the Vietoris or Fell topology.

In case of a metrizable (locally) compact space X the topology of the hyperspace $\mathrm{Cld}_V(X)$ (resp. $\mathrm{Cld}^*_F(X)$) and some of its subspaces is now well-studied. The first principal result of this sort was Curtis-Shori Theorem [CS] characterizing hyperspaces homeomorphic to the Hilbert cube. In the following theorem we unify the latter result with a result of D. Curtis and N.-T. Nhu [CN] on the topology of the pair $(\mathrm{Cld}_V(X), \mathrm{Fin}_V(X))$.

We shall say that a pair (X,Y) of topological spaces $Y\subset X$ is homeomorphic to a pair (A,B) if there is a homeomorphism $h: X\to A$ with $h(Y)=B$. A topological space X is defined to be *strongly countable-dimensional* if X is a countable union of closed finite-dimensional subsets. A topological space X is *locally connected* if each point $x \in X$ has a neighborhood base consisting of open connected neighborhoods.

Theorem 2 (Curtis-Shori [CS] and Curtis-Nhu [CN]). *The hyperspace* $\mathrm{Cld}_F(X) = \mathrm{Cld}_V(X)$ *of non-empty closed subsets of a compact space* X *is homeomorphic to the Hilbert cube* Q *if and only if* X *is metrizable, connected, locally connected, and contains more than one point. Moreover, in this case the pair* $(\mathrm{Cld}_F(X), \mathrm{Fin}_F(X))$ *is homeomorphic to* (Q, σ) *iff* X *is strongly countably dimensional.*

The non-compact case was considered by K. Sakai and Z. Yang [SY]:

Theorem 3 (Sakai-Yang [SY]). *The hyperspace* $\mathrm{Cld}^*_F(X)$ *of a locally compact space* X *is homeomorphic to the Hilbert cube* Q *if and only if the pair* $(\mathrm{Cld}^*_F(X), \mathrm{Comp}^*_F(X))$ *is homeomorphic to* (Q, Σ) *iff* X *is non-compact, metrizable, separable, locally connected, and has no compact connected component. Moreover, in this case the pair* $(\mathrm{Cld}^*_F(X), \mathrm{Fin}^*_F(X))$ *is homeomorphic to the pair* (Q, σ) *if and only if the space* X *is strongly countably dimensional.*

In [SY] K. Sakai and Z. Yang asked about the topological structure of the triple $(\mathrm{Cld}^*_F(X), \mathrm{Comp}^*_F(X), \mathrm{Fin}^*_F(X))$. We shall answer this problem by proving

Theorem 4. *Let* X *be a locally compact space. The triple* $(\mathrm{Cld}^*_F(X), \mathrm{Comp}^*_F(X), \mathrm{Fin}^*_F(X))$ *is homeomorphic to the triple* (Q, Σ, σ) *if and only if* X *is non-compact, metrizable, separable, locally connected, strongly countably dimensional, and has no compact connected component.*

In particular, when $X = \mathbb{R}^n$ we have $(\mathrm{Cld}^*_F(X), \mathrm{Comp}^*_F(X), \mathrm{Fin}^*_F(X)) \cong (Q, \Sigma, \sigma)$.

We shall present the proof of this theorem in a separate section because of its technical character.

The above theorems combined with the product structure of the hyperspace $\mathrm{Cld}^*_F(X)$ allow us to find many model non-metrizable compacta (like Tychonov cube or Cantor cube) among hyperspaces $\mathrm{Cld}^*_F(X)$.

In the following theorem for a topological space X by $X \cup \{*\}$ we denote X with attached isolated point $* \notin X$.

Theorem 5. *Let* X *be a non-compact metrizable locally compact locally connected space. Let* δ, μ, η *be the families of connected components of* X *that are degenerated, compact non-degenerated, and non-compact, respectively. Then, there is a homeomorphism* $h : \mathrm{Cld}^*_F(X) \to 2^\delta \times (Q \cup \{*\})^\mu \times Q^\eta$ *such that*

(1) $h(\mathrm{Cld}^*_{<\tau}(X)) = 2^\delta_{<\tau} \times (Q \cup \{*\})^\mu_{<\tau} \times Q^\eta_{<\tau}$ *for any uncountable cardinal* τ*;*
(2) $h(\mathrm{Comp}^*(X)) = 2^\delta_{<\omega} \times (Q \cup \{*\})^\mu_{<\omega} \times \Sigma^\eta_{<\omega}$*;*
(3) $h(\mathrm{Fin}^*(X)) = 2^\delta_{<\omega} \times (\sigma \cup \{*\})^\mu_{<\omega} \times \sigma^\eta_{<\omega}$ *provided* X *is strongly countably dimensional.*

Proof. Let $\mathcal{C} = \delta \cup \mu \cup \eta$ be the family of connected components of X and observe that $X = \sqcup \mathcal{C}$.

Define a homeomorphism $g : \mathrm{Cld}^*_F(X) \to \prod_{Y \in \mathcal{C}} \mathrm{Cld}^*_F(Y)$ as usual: assign to each closed subset $F \subset X$ the family of intersections $g(F) = (F \cap Y)_{Y \in \mathcal{C}} \in \prod_{Y \in \mathcal{C}} \mathrm{Cld}^*_F(Y)$.

For every $Y \in \delta$ the space Y is a singleton. Consequently, $\mathrm{Cld}^*_F(Y)$ is homeomorphic to the two-point space 2 by a homeomorphism $g_Y : \mathrm{Cld}^*_F(Y) \to 2$.

For every $Y \in \mu$ the space Y is compact and we can apply Theorem 2 to find a homeomorphism $f_Y : (\mathrm{Cld}^*_F(Y), \mathrm{Fin}^*_F(Y)) \to (Q \cup \{*\}, \sigma \cup \{*\})$.

Finally, for every $Y \in \eta$ the space Y is not compact and we can apply Theorem 4 to find a homeomorphism f_Y mapping the triple $(\mathrm{Cld}^*_F(Y), \mathrm{Comp}^*_F(Y), \mathrm{Fin}^*_F(Y))$ onto (Q, Σ, σ).

Let $f = \prod_{Y\in\mathcal{C}} f_Y : \prod_{Y\in\mathcal{C}} \mathrm{Cld}^*_F(Y) \to 2^\delta \times (Q\cup\{*\})^\mu \times Q^\eta$. Then the composition $h = f \circ g : \mathrm{Cld}^*_F(X) \to 2^\delta \times (Q \cup \{*\})^\mu \times Q^\eta$ is a desired homeomorphism having the properties (1)–(3). □

Corollary 5. *Let X be a metrizable locally compact space of weight $\kappa = w(X)$. The hyperspace $\mathrm{Cld}^*_F(X)$ is homeomorphic to the Tychonov cube I^κ iff X is locally connected and has no compact connected component. In this case the pair $(\mathrm{Cld}^*_F(X), \mathrm{Cld}^*_{<\tau}(X))$ is homeomorphic to $(Q^\kappa, Q^\kappa_{<\tau})$ for any cardinal τ. Moreover, the pair $(\mathrm{Cld}^*_F(X), \mathrm{Fin}^*_F(X))$ is homeomorphic to the pair $(I^\kappa, I^\kappa_{<\omega})$ provided X is strongly countably dimensional.*

Applying these results to metrizable locally compact topological groups, we get

Theorem 6. *The hyperspace $\mathrm{Cld}^*_F(X)$ of a locally connected first countable locally compact group X is homeomorphic to 2^κ, $(Q \cup \{*\})^\kappa$, or Q^κ where κ is the number of connected components of X.*

Proof. Being first countable, the group X is metrizable. By Lemma 1, X is homeomorphic to product of the form $\mathbb{R}^n \times K \times D$, where $n \in \mathbb{N} \cup \{0\}$, K is a compact group, and D is a discrete space. Since X is locally connected, so is the group K. Moreover, replacing K by the connected component of the identity we may assume that K is connected.

If $n = 0$ and K is trivial, then X is discrete and $\mathrm{Cld}^*_F(X)$ is homeomorphic to 2^κ, where $\kappa = |X| = w(X)$.

If $n = 0$ and K is not trivial, then all connected components of G are homeomorphic to K and thus are compact and non-degenerated. Consequently, $\mathrm{Cld}^*_F(X)$ is homeomorphic to $(Q \cup \{*\})^\kappa$ by Theorem 5.

If $n \neq 0$, then each connected component of X is homeomorphic to $\mathbb{R}^n \times K$ and hence is not compact. By Theorem 5, $\mathrm{Cld}^*_F(X)$ is homeomorphic to Q^κ. □

On the other hand, for locally compact zero-dimensional first countable group X the topology of $\mathrm{Cld}^*_F(X)$ also can be described. First we prove a lemma.

Lemma 3. *Let X be a locally compact σ-compact non-compact metrizable zero-dimensional space. Then the hyperspace $\mathrm{Cld}^*_F(X)$ is homeomorphic to the Cantor cube $C = 2^\omega$.*

Proof. Observe that the one-point compactification αX is zero-dimensional and metrizable and so is the hyperspace $\mathrm{Cld}_V(\alpha X)$, see [To]. Then $\mathrm{Cld}^*_F(X)$ is a zero-dimensional metrizable compact space, because it can be identified with a closed subspace of $\mathrm{Cld}_V(\alpha X)$.

Now the topological equivalence of $\mathrm{Cld}^*_F(X)$ and the Cantor cube will follow from the topological characterization of 2^ω as soon as we check that $\mathrm{Cld}^*_F(X)$ has no isolated point. Since $\mathrm{Fin}^*_F(X)$ is dense in $\mathrm{Cld}^*_F(X)$, it suffices to check that $\mathrm{Fin}^*_F(X)$ has no isolated point. Take any sequence $(x_n)_{n\in\omega}$ of pairwise distinct points of X tending to infinity. Then for any finite subset $A \subset X$ the sequence of finite sets $(A\cup\{x_n\})_{n\in\omega}$ tends to A in $\mathrm{Fin}^*_F(X)$. Also $A\cup\{x_n\} \neq A$ for sufficiently large n. □

Theorem 7. *The hyperspace* $\mathrm{Cld}^*_F(X)$ *of a zero-dimensional first countable locally compact group* X *is homeomorphic to* $(2^\omega \cup \{*\})^\kappa$ *or* 2^κ *for some cardinal* κ.

Proof. If X is finite of size k, then $\mathrm{Cld}^*_F(X)$ is homeomorphic to the discrete cube 2^κ.

If X is compact and infinite, then it is homeomorphic to the Cantor cube 2^ω by Brouwer's Theorem, which implies $\mathrm{Cld}^*_F(X) \cong 2^\omega \cup \{*\}$.

If X is not compact, then $X = \sqcup X_{i\in\kappa}$ is a disjoint sum of clopen σ-compact non-compact subsets X_i of X for some cardinal κ. Then, by Proposition 2 the hyperspace $\mathrm{Cld}^*_F(X)$ is homeomorphic to the product $\prod_{i\in\kappa} \mathrm{Cld}^*_F(X_i)$ and by Lemma 3, each hyperspace $\mathrm{Cld}^*_F(X_i)$ is homeomorphic to 2^ω. Consequently, $\mathrm{Cld}^*_F(X)$ is homeomorphic to $(2^\omega)^\kappa$. □

4. Proof of Theorem 4

First, we give preliminary notions and results of infinite-dimensional topology used in the proof of Theorem 4. Let X be a separable metrizable space. A set $A \subset X$ is defined to be a *Z-set* provided it is closed and for every $\mathcal{U} \in cov(X)$ there exists a map $f : X \to X$ such that $(f, id) \prec \mathcal{U}$ and $f(X) \cap A = \emptyset$. By $cov(X)$ we denote the set of all open covers. Two maps are called $\mathcal{U}$-near (denoted by $(f, g) \prec \mathcal{U}$) if the family $\{f(y), g(y)\}_{y\in Y}$ is inscribed into $\mathcal{U}$. We abbreviate "absolute neighborhood retract" to "ANR". For other details, we refer to the book [BRZ].

By a *tower* in a space X we understand any increasing sequence $X_1 \subset X_2 \subset \ldots \subset X_n \subset \ldots \subset X$ of subsets of X. A tower $(X_n)_{n\in\mathbb{N}}$ of subsets of a space X is defined to satisfy

- *the flattening property in* X if given a cover $\mathcal{U} \in cov(X)$ there is a map $f : X \to X$ such that $(f, id) \prec \mathcal{U}$ and every point $x \in X$ has a neighborhood $W \subset X$ with $W \cap f(X) \subset X_n$ for some $n \in \mathbb{N}$;
- *the strong flattening property in* X if for every open set $U \subset X$ the tower $(U \cap X_n)_{n\in\mathbb{N}}$ has the flattening property in U.

The proof of Theorem 4 rely on the following technical results.

Lemma 4. [SW] *A triple* (X, Y, Z) *is homeomorphic to* (Q, Σ, σ) *if and only if:*

1. *X is a compact absolute retract;*
2. *Y is a σZ-set in X;*
3. *$Z \subset Y$ is σ-compact and strongly countable dimensional;*
4. *(X, Y, Z) is strongly universal for the class of triples (A, B, C), where $A = B$ is a compact and $C \subset B$ is strongly countably dimensional and σ-compact.*

The next result belongs to T. Banakh [Ba, Theorem 4.1]:

Lemma 5. *Let X be a locally compact ANR, and (X, Y, Z), (A, B, C) be two triples. The triple (X, Y, Z) is strongly (A, B, C)-universal, provided there exists a tower $X_1 \subset X_2 \subset \ldots \subset X$ of Z-sets such that*

1. *for every $i \in \mathbb{N}$ X_i is an ANR such that the triple $(X_i \cap X, X_i \cap Y, X_i \cap Z)$ is strongly (A, B, C)-universal;*
2. *the tower $(X_i)_{i\in\mathbb{N}}$ has the strong flattening property.*

We also need the following decomposition result:

Lemma 6. *Let X be a non-compact, locally compact, separable, metrizable, connected and locally connected space. Then, X can be presented as $X = \bigcup_{i\in\mathbb{N}} X_i$, where $X_i \subset \operatorname{int}(X_{i+1})$ (for $A \subset X$ by $\operatorname{int}(A)$ we denote its interior in X), and X_i is a Peano continuum* [1] *for each $n \in \mathbb{N}$.*

Proof. First, we show that each point $x \in X$ has a connected locally connected compact neighborhood. Indeed, $\mathrm{Comp}_V(X)$ is homeomorphic to $Q \setminus \{*\}$ by [Cu$_1$], hence each $\{x\} \in \mathrm{Cld}^*_F(X)$ has a neighborhood $N(\{x\})$ with compact closure homeomorphic to the Hilbert cube Q. Then, by [CN, Lemma 2.2] $\bigcup \bar{N}(\{x\})$ is a neighborhood of $x \in X$ with the required properties. Inductively, we shall construct a tower of Peano continua $(X_n)_{n\in\mathbb{N}}$ such that $X_n \subset \operatorname{int}(X_{n+1})$ and $X = \bigcup_{n\in\mathbb{N}} X_n$. Choose any point $x \in X$ and find a compact connected locally connected neighborhood $\bar{O}(x)$, and define $X_1 = \bar{O}(x)$. Assume that X_{i-1} have been constructed. There exists a cover of X_{i-1} by connected locally connected neighborhoods $\mathcal{U} = \{O(x)\}_{x\in X_{i-1}}$ with compact closures. Since X_{i-1} is compact, find a finite subcover $\mathcal{F} = \{O(x_1), \dots, O(x_m)\}$ of $\mathcal{U}$ and define $X_i = \bar{O}(x_1) \cup \ldots \cup \bar{O}(x_m)$. Note that X_i is a Peano continuum and $X_{i-1} \subset \operatorname{int}(X_i)$. The inductive step is over. So, by induction we build a tower as required. □

Proof of Theorem 4. First, we consider the case of connected X.

The "only if" part follows from Theorems 2 and 3.

The "if" part. We use Lemma 4. The items (1), (2), (3) of Lemma 4 follow from Theorem 3. It remains to show that the triple $(\mathrm{Cld}^*_F(X), \mathrm{Comp}^*_F(X), \mathrm{Fin}^*_F(X))$ is strongly universal for the class of triples defined in Lemma 4. Using Lemma 6 we define a tower $(\mathrm{Comp}_F(X_i))_{i\in\mathbb{N}}$ in $\mathrm{Cld}^*_F(X)$, where each X_i is a Peano continuum. Since $\mathrm{Cld}^*_F(X) \cong Q$ by Theorem 3, it is a compact absolute retract. Then, according to Lemma 5 we should prove that

1. for every $i \in \mathbb{N}$ $\mathrm{Comp}_F(X_i)$ is a Z-set in $\mathrm{Cld}^*_F(X)$ and ANR such that the triple $(\mathrm{Comp}_F(X_i), \mathrm{Comp}_F(X_i), \mathrm{Fin}_F(X_i))$ is strongly universal for the class of triples defined in Lemma 4, or equivalently $(\mathrm{Comp}_F(X_i), \mathrm{Fin}_F(X_i))$ is strongly universal for the class of pairs (B, C), where B is a compact and C is strongly countable dimensional and σ-compact;
2. the tower $\mathrm{Comp}_F(X_i)$ has the strong flattening property in $\mathrm{Cld}^*_F(X)$.

1. First, we prove that $\mathrm{Comp}_F(X_i)$ is a Z-set in $\mathrm{Cld}^*_F(X)$ for every $i \in \omega$. We should show that, see [BRZ, Theorem 1.4.4], for every $\varepsilon > 0$, $n \in \omega$, and every map $f : I^n \to \mathrm{Cld}^*_F(X)$ there is a map $g : I^n \to \mathrm{Cld}^*_F(X)$ such that $d(f, g) < \varepsilon$ and $g(I^n) \cap \mathrm{Comp}_F(X_i) = \emptyset$. Let f be such a map and $\varepsilon > 0$. Since $\mathrm{Cld}^*_F(X)$ is a compact Lawson semilattice (as a subsemilattice of the semilattice $\mathrm{Cld}_V(\alpha X)$, see the proof of Proposition 3), the union operator $\cup : \mathrm{Cld}^*_F(X)^2 \to \mathrm{Cld}^*_F(X)$ is uniformly continuous (for the Lawson semilattices see [CoK]). Whence, we can choose a compatible metric d on $\mathrm{Cld}^*_F(X)$ such that for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ with the following property: for each $x \in X$ and $F \in \mathrm{Cld}^*_F(X)$ if $d(\{x\}, \{\emptyset\}) < \delta$ then $d(F \cup \{x\}, F) < \varepsilon$. Then, choose a point $x_0 \in X$ such that $x_0 \notin X_i$ and $d(\{x_0\}, \{\emptyset\}) < \delta(\varepsilon)$. Whence, it is easy to see that the map $g = f \cup \{x_0\}$ is as required.

Note, that $\mathrm{Comp}_F(X_i)$, $i \in \omega$, is homeomorphic to Q by Theorem 2, and so is an absolute retract.

[1] A Peano continuum is a connected locally connected compact metrizable space.

Finally, by Theorem 2, the pair $(\mathrm{Comp}_F(X_i), \mathrm{Fin}_F(X_i))$ is homeomorphic to the pair (Q, σ), which is strongly universal for the class of pairs defined in the item (1).

2. Since X is σ-compact and strongly countably-dimensional, X is a countable union of finite-dimensional compact sets. As is shown in [Cu$_3$, Continuation of proof of (3.2)], X has a tower $(\Gamma_i)_{i\in\mathbb{N}}$ of finite connected graphs such that the tower $\mathrm{Fin}_i(\Gamma_i)_{i\in\mathbb{N}} = \{F \in \Gamma_i : |F| \le i\}$ has the mapping approximation property in $\mathrm{Cld}^*_F(X)$ which is equivalent to the mapping absorption property for finite-dimensional compacta by [Cu$_2$, Theorem 4.6]. Then, the tower $\mathrm{Fin}_i(\Gamma_i)_{i\in\mathbb{N}}$ enjoys the strong flattening property by [Ba, Proposition 3.2]. Whence, the tower $(\mathrm{Comp}^*_F(X_i))_{i\in\mathbb{N}}$ has the strong flattening property as well, since it contains the tower $\mathrm{Fin}_i(\Gamma_i)_{i\in\mathbb{N}}$ (see the definition of the strong flattening proprety).

So, we have the item (2), and theorem is proved for the connected case.

Now, consider the disconnected case. By the similarity, we give a proof only when X has infinitely many components. Let $X = \bigcup_{n\in\omega} X_n$, where X_n's are connected components of X. By the above connected case,

$$(\mathrm{Cld}^*_F(X_n), \mathrm{Comp}^*_F(X_n), \mathrm{Fin}^*_F(X_n)) \cong (Q, \Sigma, \sigma),\ n \in \omega.$$

On the other hand, we have a homeomorphism

$$\xi : \mathrm{Cld}^*_F(X) \to \prod_{n\in\omega} \mathrm{Cld}^*_F(X_n)$$

defined by $\xi(A) = (A \cap X_n)_{n\in\omega}$, whence $\xi^{-1}(A_i)_{i\in\omega} = \bigcup_{i\in\omega} A_i$. Observe

$$\xi(\mathrm{Comp}^*(X)) = \{(A_n)_{n\in\omega} | \forall n \in \omega,\ A_n \in \mathrm{Comp}^*(X_n) \text{ and } A_n = \emptyset \\ \text{except for finitely many } n \in \omega\} \subset \prod_{n\in\omega} \mathrm{Comp}^*_F(X_n) \subset \prod_{n\in\omega} \mathrm{Cld}^*_F(X_n)$$

and

$$\xi(\mathrm{Fin}^*(X)) = \{(A_n)_{n\in\omega} | \forall n \in \omega,\ A_n \in \mathrm{Fin}^*(X_n) \text{ and } A_n = \emptyset \\ \text{except for finitely many } n \in \omega\} \subset \prod_{n\in\omega} \mathrm{Fin}^*_F(X_n) \subset \prod_{n\in\omega} \mathrm{Cld}^*_F(X_n).$$

Then, it follows from the connected case that $(\mathrm{Cld}^*_F(X), \mathrm{Comp}^*_F(X), \mathrm{Fin}^*_F(X)) \cong (Q^\omega, \Sigma^\omega_{<\omega}, \sigma^\omega_{<\omega}) \cong (Q, \Sigma, \sigma)$, see [SY].

This completes the proof of theorem. □

5. Concluding discussion and open questions

Remark 1. *The paracompactness is essential in Theorem 1. Indeed, consider the ordinal space $X = [0, \omega_1)$ endowed with the order topology. This space is locally compact and locally countable. Consequently, $lc(X) = ld(X) = \aleph_0$. On the other hand, $c(\mathrm{Cld}^*_F(X)) = \aleph_1$. Indeed, to each non-limit ordinal $\alpha < \omega_1$ assign the open set $U_\alpha = \{F \in \mathrm{Cld}^*_F(X) : \alpha \in F \text{ and } F \cap [0, \alpha) = \emptyset\}$ in $\mathrm{Cld}^*_F(X)$. Then $(U_\alpha)_\alpha$ is an uncountable family of pairwise disjoint open sets in $\mathrm{Cld}^*_F(X)$, which shows that $\aleph_1 \le c(\mathrm{Cld}^*_F(X)) \le sh(\mathrm{Cld}^*_F(X)) \le d(\mathrm{Cld}^*_F(X)) \le w(\mathrm{Cld}^*_F(X)) = \aleph_1$.*

Let us also remark that the ordinal space $[0, \omega_1)$ is locally metrizable but not metrizable. As we saw in Proposition 2, the hyperspace $\mathrm{Cld}^*_F(X)$ of a metrizable locally compact space X is a product of metrizable compacta.

There are some interesting properties lying in between the countable cellularity and being a product of metrizable compacta. We recall that a compact Hausdorff space is defined to be

- an AE(0)-*space* if any continuous map $f : B \to X$ defined on a closed subset B of a zero-dimensional compact space Z admits a continuous extension $\bar{f} : Z \to X$;
- *scattered compact* if each subspace of X contains an isolated point;
- a *dyadic space* if X is a continuous image of a Cantor cube $\{0,1\}^\kappa$ for some cardinal κ;
- a *polyadic space* if X is a continuous image of a product of one-point compactifications of discrete spaces;
- a *scadic space* if X is a continuous image of a product of scattered compact spaces;
- a *hyadic space* if X is a continuous image of the hyperspace $\mathrm{Cld}_V(X)$ of some compact Hausdorff space.

These properties relate as follows, see [Bell]:

$$\begin{array}{ccccccccc}
\begin{pmatrix}\text{product}\\ \text{of metric}\\ \text{compacta}\end{pmatrix} & \to & \mathrm{AE}(0) & \to & (\text{dyadic}) & \to & sh(X)\le\aleph_0 & \to & c(X)\le\aleph_0\\
 & & & & \downarrow & & & & \\
 & & & & (\text{polyadic}) & & & & \\
 & & & \swarrow & & \searrow & & & \\
 & & & (\text{scadic}) & & (\text{hyadic}) & & &
\end{array}$$

Proposition 3. *For any locally compact space X the space $\mathrm{Cld}^*_F(X)$ is a hyadic space.*

Proof. This follows from the fact that the hyperspace $\mathrm{Cld}^*_F(X)$ is homeomorphic to the retract $\{F \in \mathrm{Cld}_V(\alpha X) : \infty \in F\}$ of the hyperspace $\mathrm{Cld}_V(\alpha X)$ over the one-point compactification $\alpha X = X \cup \{\infty\}$. □

Remark 2. *In Example 8 of* [Ho] *it was noticed that the hyperspace $\mathrm{Cld}_F(X)$ over the Alexadrov "double arrow" space X is first countable but not metrizable and consequently $\mathrm{Cld}_F(X)$ fails to be dyadic. In fact, for the "double arrow" space X the hyperspace $\mathrm{Cld}_F(X)$ fails to be scadic because the weight of each scadic compact space coincides with its character, see* [Bell].

The other properties from the diagram pose open problems.

Question 1. *Is there a locally compact space X whose hyperspace $\mathrm{Cld}^*_F(X)$ is (scadic but) not polyadic? Polyadic but not dyadic? Has countable Suslin number but uncountable Shanin number? Has countable Shanin number but is not dyadic? Is dyadic but not AE(0)-space?*

It is known that a compact Hausdorff space X has weight $w(X) \le \aleph_1$ if its hyperspace $\mathrm{Cld}_V(X) = \mathrm{Cld}_F(X)$ is a dyadic space, see [Sha].

Question 2. *Let X be a locally compact space such that the hyperspace $\mathrm{Cld}^*_F(X)$ is dyadic (is an AE(0)-compactum). Is the local weight $lw(X) \le \aleph_1$? Is X paracompact?*

It is important to remark that a locally compact space X with AE(0) hyperspace $\mathrm{Cld}^*_F(X)$ need not be metrizable.

Example 1. *The hyperspace* $\mathrm{Cld}^*_F(X)$ *of the space* $X = \omega \times 2^{\omega_1}$ *is homeomorphic to* $(2^{\omega_1} \cup \{*\})^\omega$ *and thus is an AE(0)-space which fails to be a product of metrizable spaces.*

Proof. One should note that X can be presented as $\bigsqcup_{i\in\omega} 2^{\omega_1}$. Thus, by Proposition 2 $\mathrm{Cld}^*_F(X)$ is homeomorphic to $\prod_{i\in\omega} \mathrm{Cld}^*_F(2^{\omega_1})$. Since the Fell topology coincides with the Vietoris topology when X is compact, and $\mathrm{Cld}^*_V(2^{\omega_1})$ by Sirota's theorem (see [To, Section 26]) is homeomorphic to $2^{\omega_1} \cup \{*\}$, we have that $\mathrm{Cld}^*_F(X)$ is homeomorphic to $(2^{\omega_1} \cup \{*\})^\omega$. □

Remark 3. *The hyperspace* $\mathrm{Cld}^*_F(X)$ *of the space* $X = \omega \times 2^{\omega_1}$ *is not homeomorphic to* 2^{ω_1} *because* $\mathrm{Cld}^*_F(X)$ *is first countable at the empty set. By analogy, the hyperspace* $\mathrm{Cld}^*_F(X)$ *of the space* $X = \omega \times I^{\omega_1}$ *is not homeomorphic to* I^{ω_1}.

Remark 4. *The hyperspace of the non-metrizable space* $X = \omega_1 \times 2^{\omega_1}$ *is homeomorphic to the Cantor cube* $2^{\omega_1} \cong (2^{\omega_1} \cup \{*\})^{\omega_1}$.

Question 3. *Is a locally compact space* X *metrizable if* $\mathrm{Cld}^*_F(X)$ *is homeomorphic to the Tychonov cube?*

Remark 5. *By* [Sha] *each non-metrizable compact space* X $\mathrm{Cld}_F(X)$ *is not a continuous image of a Tychonov cube.*

References

[Ar] A.V. Arkhangelskiĭ, *Structure and classification of topological spaces and cardinal invariants*, Uspekhi Mat. Nauk, **33**:6 (1978), 29–64 (in Russian).

[Ba] T. Banakh, The strongly universal property in convex sets, Matem. Studii, **10** (1998), 203-218.

[Bell] M. Bell, *On character and chain conditions in images of products*, Fundam. Math. **158**:1 (1998), 41-49.

[Be] G. Beer, *Topologies on Closed and Closed Convex Sets*, Kluwer Academic, Dordrecht, 1993.

[BRZ] T. Banakh, T. Radul and M. Zarichnyi, *Absorbing Sets in Infinite-Dimensional Manifolds*, VNTL Publishers, Lviv, 1996.

[CoK] C. Costantini, W. Kubiś, *Paths in hyperspaces*, Appl. Gen. Topol. **4** (2003), 377-390.

[CN] Doug Curtis and Nguyen To Nhu, *Hyperspaces of finite subsets which are homeomorphic to* $\aleph_0$*-dimensional linear metric spaces*, Topol. Appl. **19** (1985), 251-260.

[CS] D.W. Curtis and R.M. Schori, *Hyperspaces of Peano continua are Hilbert cubes*, Fund. Math. **101** (1978), 19-38.

[Cu$_1$] D.W. Curtis, *Hyperspaces of non-compact metric spaces*, Comp. Math. **40** (1980), 139-152.

[Cu$_2$] D.W. Curtis, *Boundary sets in the Hilbert cube*, Topol Appl. **20** (1985), 201-221.

[Cu$_3$] D.W. Curtis, *Hyperspaces of finite subsets as boundary sets*, Topol Appl. **22** (1986), 97-107.

[En] R. Engelking, *General Topology*. PWN, Warszawa, 1977.

[Fe] J.M.G. Fell, *A Hausdorff topology for the closed subsets of a locally compact non-Hausdorff space*, Proc. Amer. Math. Soc. **13** (1962), 472-476.

[FF] V.V. Fedorchuk, V.V. Filippov, *General topology. Basic constructions*, Moscow Univ. Press, Moscow. (in Russian).

[FT] V. Fedorchuk, S. Todorčevič. *Cellularity of covariant functors*, Topology Appl. **76**:2 (1997), 125-150.

[He] E. Hewitt and K.A. Ross, *Abstract Harmonic Analysis, Vol I.*, Springer-Verlag, 1975.

[Ho] Hou, Ji-Cheng, *Character and tightness of hyperspaces with the Fell topology*, Topology Appl. **84** (1998), 199-206.

[IN] A. Illanes, S. Nadler, *Hyperspaces: fundamentals and recent advances*, PAM 216. NY: Marcel Dekker, 1999.

[Pro$_1$] I. Protasov, *The Suslin number of a space of subgroups of a locally compact group*, Ukr. Mat. Zh. 40:5 (1988), 654-658.

[Pro$_2$] I. Protasov, *Personal communication*.

[Sha] L. Shapiro, *Spaces of closed subsets of bicompacta*, Soviet Math. Dokl. 17 (1976), no. 3, 937941.

[SW] K. Sakai and R.Y. Wong, *On infinite-dimensional manifold triples*, Trans. Amer. Math. Soc. **318** (1990), 545-555.

[SY] K. Sakai, Z. Yang, *Hyperspaces of non-compact metrizable spaces which are homeomorphic to the Hilbert cube*, Topology Appl. **127** (2003), 331-342.

[To] S. Todorčevič, *Topics in topology*, Springer-Verlag, 1997.

[TZ] A. Teleiko, M. Zarichnyi, *Categorial topology of compact Hausdorff spaces*, VNTL Publishers, Lviv, 1999.

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