A UNIVERSAL MODEL INFINITE-DIMENSIONAL SPACE

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Abstract. Given an ordinal α and a pointed topological space X, we endow $X^{<\alpha} = \cup\{X^\beta : \beta < \alpha\}$ with the strongest topology that coincides with the product topology on every subset X^β of $X^{<\alpha}$, $\beta < \alpha$. It turns out that many important model spaces of infinite-dimensional topology (including the topology of nonmetrizable manifolds) can be obtained as spaces of the form $X^{<\alpha}$ for $X = I, \mathbb{R}$. The paper deals with some topological properties of spaces $X^{<\alpha}$. Some new classification and characterization theorems are proved for these spaces.

1. Introduction

A considerable part of the classical infinite-dimensional topology deals with manifolds modeled on some nice model infinite-dimensional spaces. Among the most important model spaces let us mention the Hilbert cube $Q = [-1,1]^\omega$, the countable product of lines $\mathbb{R}^\omega$, the Tychonov cube I^τ, the uncountable power of the line $\mathbb{R}^\tau$, the direct limit $\mathbb{R}^\infty$ of Euclidean spaces and the direct limit Q^∞ of Hilbert cubes. The topological characterizations of these model spaces can be found in [To1], [To2], [Chi], [FC], [S], [Sa] and are among the most prominent achievements of the classical infinite-dimensional topology.

It turns out that all these model spaces are particular examples of one fairly general topological construction we are going to describe now.

We shall identify cardinals with initial ordinals of a given size. Each ordinal α will be identified with the set of all ordinals $< \alpha$. By a pointed space we understand a topological space X with some distinguished point $*$ of X. In the sequel we shall consider the real line $\mathbb{R}$ and the interval $I = [-1,1]$ as pointed spaces whose distinguished point is zero. The distinguished point of a Tychonov cube I^τ is the constant zero function.

Given two ordinals $\beta < \alpha$ and a pointed topological space X with a distinguished point $*$ identify the power X^β with the subset $\{(x_i)_{i\in\alpha} \in X^\alpha : x_i = * \text{ for all } i \geq \beta\}$. Let

$$X^{<\alpha} = \bigcup_{\beta<\alpha} X^\beta$$

and endow the space $X^{<\alpha}$ with the strongest topology inducing the product topology on each subset $X^\beta \subset X^{<\alpha}$, $\beta < \alpha$. We shall refer to this topology on $X^{<\alpha}$ as the *strong topology* in contrast to the *product topology*. In infinite-dimensional topology the spaces of the form $X^{<\omega}$ usually are denoted by X^∞. For some special pointed spaces X like the closed interval $I = [-1,1]$, the real line $\mathbb{R}$, Hilbert cube $Q = I^\omega$ or the Hilbert space l_2, the spaces $X^\infty = X^{<\omega}$ were topologically characterized in [Sa] and [Pe].

For such particular X the spaces $X^{<\alpha}$ yield us almost all known model spaces of the classical infinite-dimensional topology. Namely, the space $I^{<\alpha}$ coincides with

- the n-dimensional cube if $\alpha = n$;
- the direct limit $I^\infty = \varinjlim I^n$ of finite-dimensional cubes (homeomorphic to $\mathbb{R}^\infty$) if $\alpha = \omega$;
- the Hilbert cube Q if $\alpha = \omega + 1$;
- the direct limit $Q^\infty = \varinjlim Q^n$ of Hilbert cubes if $\alpha = \omega \cdot \omega$;
- a non-metrizable Tychonov cube I^τ if $\alpha = \tau + 1$ is uncountable successor ordinal;
- the direct limit $(I^\tau)^\infty = \varinjlim (I^\tau)^n$ of Tychonov cubes if $\alpha = \tau \cdot \omega$;

1991 *Mathematics Subject Classification.* 57N20; 54B10.
Key words and phrases. Hilbert cube, Tychonov cube.

- the Σ-product $\Sigma(I)=\{f\in I^{\omega_1}:|\{\alpha\in\omega_1:f(\alpha)\ne 0\}|\le\omega\}\subset I^{\omega_1}$ of intervals if $\alpha=\omega_1$ (as we shall see in Coincidence Theorem 2.2, the strong topology on $I^{<\omega_1}$ coincides with the product topology).

On the other hand, the spaces $\mathbb{R}^{<\alpha}$ yield us

- the Euclidean space $\mathbb{R}^n$ if $\alpha=n$;
- the direct limit $\mathbb{R}^\infty=\varinjlim\mathbb{R}^n$ of Euclidean spaces if $\alpha=\omega$;
- the countable product of lines $\mathbb{R}^\omega$ (homeomorphic to the separable Hilbert space l_2) if $\alpha=\omega+1$;
- the direct limit $(\mathbb{R}^\omega)^\infty=\varinjlim(\mathbb{R}^\omega)^n$ if $\alpha=\omega\cdot\omega$ (the latter space is homeomorphic to the direct limit of Hilbert spaces $(l_2)^\infty$ and was studied by E.Pentsak [Pe]);
- an uncountable product of lines $\mathbb{R}^\tau$ if $\alpha=\tau+1$ is an uncountable successor ordinal;
- the Σ-product $\Sigma(\mathbb{R})=\{f\in\mathbb{R}^{\omega_1}:|\{\alpha\in\omega_1:f(\alpha)\ne 0\}|\le\omega\}$ of the lines if $\alpha=\omega_1$.

Thus the spaces of the form $X^{<\alpha}$ can be considered as universal model spaces for infinite-dimensional topology. In this paper we shall be interested in three general problems concerning these spaces:

1. Investigate topological properties of the spaces $X^{<\alpha}$ for various ordinals α.
2. Give a topological classification of the spaces $X^{<\alpha}$.
3. Find topological characterizations of the spaces $X^{<\alpha}$ for simple spaces X (like I or $\mathbb{R}$) and simple ordinals α.

2. Survey of principal results

We start the investigation of the spaces $X^{<\alpha}$ with calculating some of their cardinals characteristics.

By a *k-space* we understand a Hausdorff topological space X admitting a cover $\mathcal{K}$ by compact subspaces, generating the topology of X in the sense that a subset $U\subset X$ is open in X if and only if for any compactum $K\in\mathcal{K}$ the intersection $U\cap K$ is open in K, see [En]. The smallest possible size $|\mathcal{K}|$ of such a cover $\mathcal{K}$ is called the *k-ness* of X and is denoted by $\mathrm{k}(X)$, see [vD]. The k-ness of a topological space does not exceed the *compact covering number* $\mathrm{kc}(X)$ equal to the smallest size of a cover of X by compact subspaces. The *network weight* $\mathrm{nw}(X)$ of a topological space X is the smallest size $|\mathcal{N}|$ of a collection $\mathcal{N}$ of subsets of X such that for any open set $U\subset X$ and any point $x\in U$ there is an element $N\in\mathcal{N}$ with $x\in N\subset U$. For two cardinals κ,τ by $\kappa\times\tau$ we denote their product (as cardinals).

By the *cofinality* $\mathrm{cf}(\alpha)$ of an ordinal α we understand the smallest size $|C|$ of a cofinal subset $C\subset\alpha$ (the latter means that for each $x<\alpha$ there is $y\in C$ with $x\le y$).

Proposition 2.1. *For any pointed compact Hausdorff space X with $|X|>1$ and any ordinal α the space $X^{<\alpha}$ is a k-space with* $\mathrm{kc}(X^{<\alpha})=\mathrm{k}(X^{<\alpha})=\mathrm{cf}(\alpha)$ *and* $\mathrm{nw}(X^{<\alpha})=\mathrm{nw}(X)\times|\alpha|$.

Let us observe that the strong topology on $X^{<\alpha}$ coincides with the product topology if α is a successor cardinal. Surprisingly enough but the same is true also for certain limit ordinals. To characterize such ordinals we need to introduce the notion of the irreducible tail $\mathrm{tl}(\alpha)$ of an ordinal α. By definition, the *irreducible tail* $\mathrm{tl}(\alpha)$ of α is the smallest ordinal β for which there exists an ordinal $\gamma<\alpha$ such that $\alpha=\gamma+\beta$. Let us observe that $\mathrm{cf}(\alpha)\le\mathrm{tl}(\alpha)\le\alpha$; and $\mathrm{cf}(\alpha)=\mathrm{tl}(\alpha)=1$ if and only if α is a successor ordinal.

Let us also note that $\mathrm{tl}(\alpha)=\alpha$ if and only if α is *additively indecomposable* in the sense that $\beta+\gamma<\alpha$ for any $\beta,\gamma<\alpha$. In particular, the ordinal $\mathrm{tl}(\alpha)$ is additively indecomposable.

Theorem 2.2 (Coincidence Theorem)**.** *Let X be a pointed (compact Hausdorff first countable) T_1-space with $|X|>1$. For an ordinal α the strong and product topologies on $X^{<\alpha}$ coincide (if and) only if* $\mathrm{tl}(\alpha)$ *is a cardinal with* $\mathrm{cf}(\mathrm{tl}(\alpha))\ne\omega$.

This theorem implies that the strong and product topologies coincide on $I^{<\omega_1}$ but differ on $(I^{\omega_1})^{<\omega_1}$. Also for any an ordinal α with $\mathrm{cf}(\alpha)=\omega$ and any pointed space X with non-isolated distinguished point the strong topology on $X^{<\omega}$ differs from the product topology. For such ordinals α the spaces $X^{<\alpha}$ occupy a special place in the whole theory and have especially nice topological properties.

A topological space X is called a *k_ω-space* if X is a k-space with $\mathrm{k}(X) \le \omega$. k_ω-Spaces often appear in topological algebra and have many nice properties, see [FST]. In particular, they are real complete. A topological space X is called *real complete* if it is homeomorphic to a closed subspace of $\mathbb{R}^\kappa$ for some cardinal κ. Real complete spaces admit also an inner description: a Tychonov space X is real complete if any point $x \in \beta X \setminus X$ in the remainder of the Stone-Čech compactification βX of X lies in a G_δ-subset of βX missing the set X, see [En, §3.11].

Let us call a topological space X an *absolute extensor for compact spaces in dimension 0* (briefly AE(0)) if any continuous map $f : B \to X$ defined on a closed subset B of a zero-dimensional compact Hausdorff space A admits a continuous extension $\bar f\colon A \to X$ onto the whole compactum A. Removing the dimensional restrictions we get the definition of an absolute extensor (briefly AE). A space X is called an *absolute retract* (briefly an AR) it is a compact Hausdorff AE. It is well known that a compact space is an AR if it is a retract of a Tychonov cube. In particular, Tychonov cubes are absolute retracts.

Now we show that many natural topological properties of the spaces $X^{<\alpha}$ are equivalent to the countable cofinality of α.

Theorem 2.3. *For an ordinal α the following conditions are equivalent:*

1. $\mathrm{cf}(\alpha) \le \omega$*;*
2. $X^{<\alpha}$ *is a k_ω-space for any pointed compact Hausdorff space X;*
3. $X^{<\alpha}$ *is real complete for any real complete pointed space X;*
4. $X^{<\alpha}$ *is real complete for some pointed T_1-space X containing more than one point.*
5. $X^{<\alpha}$ *is an AE for any pointed absolute extensor X;*
6. $X^{<\alpha}$ *is an AE(0) for some pointed T_1-space X with $|X| > 1$.*

Now we shall discuss the topological classification of spaces $X^{<\alpha}$.

Theorem 2.4 (Reduction Theorem)**.** *For a pointed space X and an infinite ordinal α the space $X^{<\alpha}$ is homeomorphic to:*

$$\begin{cases} X^{|\alpha|} & \text{if } \alpha \text{ is a successor ordinal;}\\ X^{<|\alpha|} & \text{if } \alpha = |\alpha| \text{ is a cardinal;}\\ X^{<|\alpha|+\mathrm{cf}(\alpha)} & \text{if } 1 < \mathrm{cf}(\alpha) = \mathrm{tl}(\alpha) < |\alpha|;\\ X^{<|\alpha|+|\mathrm{tl}(\alpha)|\cdot\mathrm{cf}(\alpha)} & \text{if } 1 < \mathrm{cf}(\alpha) < \mathrm{tl}(\alpha) < |\alpha|;\\ X^{<|\alpha|\cdot\mathrm{cf}(\alpha)} & \text{if } 1 < \mathrm{cf}(\alpha) < |\mathrm{tl}(\alpha)| = |\alpha| < \alpha. \end{cases}$$

This theorem can be proved using coordinate permutating homeomorphisms and is left to the reader. Observe that the set $X^{<\alpha+\beta\cdot\gamma}$ can be naturally identified with the product $X^\alpha \times (X^\beta)^{<\gamma}$. For compact Hausdorff X this identification is topological.

Proposition 2.5. *Let X be a pointed topological space and α, β be ordinals.*

1. *The space $X^{<\alpha\cdot\beta}$ is naturally homomorphic to the space $(X^\alpha)^{<\beta}$.*
2. *If X is compact and Hausdorff, then $X^{\alpha+\beta}$ is naturally homeomorphic to the product $X^\alpha \times X^{<\beta}$.*

Remark 1. It is interesting to notice that the second statement of this proposition does not hold for non-compact spaces X. In particular, the space $\mathbb{R}^{<\omega+\omega}$ is not homeomorphic to $\mathbb{R}^\omega \times \mathbb{R}^{<\omega}$ since the former space is a k-space while the latter is not, see [Ba2].

Proposition 2.5 and Reduction Theorem 2.4 allows us to reduce the study of spaces $X^{<\alpha}$ for compact spaces X to studying the particular cases when α is a cardinal.

Corollary 2.6. *For a pointed compact space X and an ordinal α the space $X^{<\alpha}$ is homomorphic to one of the spaces: X^τ, $X^{<\tau}$, $(X^\tau)^{<\lambda}$, $X^\tau\times X^{<\lambda}$, $X^\tau\times(X^\kappa)^{<\lambda}$, where $\tau=|\alpha|$, $\lambda=\mathrm{cf}(\alpha)$, $\kappa=|\mathrm{tl}(\alpha)|$.*

For two ordinals $\alpha\ge\beta$ by $\alpha-\beta$ we denote the unique ordinal γ such that $\alpha=\beta+\gamma$. The Reduction Theorem 2.4 allows us to prove the following

Theorem 2.7 (Classification Theorem)**.** *Let X be a pointed metrizable separable space containing more than one point. For two infinite ordinals α,β the spaces $X^{<\alpha}$ and $X^{<\beta}$ are homeomorphic if and only if $|\alpha|=|\beta|$, $\mathrm{cf}(\alpha)=\mathrm{cf}(\beta)$ and $|\mathrm{tl}(\alpha)-\mathrm{cf}(\alpha)|=|\mathrm{tl}(\beta)-\mathrm{cf}(\beta)|$.*

For metrizable ARs X studying the topology of the spaces $X^{<\alpha}$ can be reduced to investigating the spaces $I^{<\alpha}$.

Theorem 2.8. *For any pointed compact metrizable absolute retract X and any ordinal $\alpha>\omega$ the space $X^{<\alpha}$ is homeomorphic to the space $I^{<\alpha}$. In its turn the space $I^{<\alpha}$ is homeomorphic to one of the spaces: I^τ, $I^{<\tau}$, $(I^\tau)^{<\lambda}$, $I^\tau\times I^{<\lambda}$, or $I^\tau\times(I^\kappa)^{<\lambda}$, where $\tau=|\alpha|$, $\lambda=|\mathrm{cf}(\alpha)|$, and $\kappa=|\mathrm{tl}(\alpha)|$.*

This theorem can be easily deduced from Corollary 2.6 and a result of H.Toruńczyk [To1] asserting that the countable power of a non-degenerate metrizable AR is homeomorphic to the Hilbert cube I^ω.

Finally we consider the problem of topological characterization of the spaces $I^{<\alpha}$. In case of countable cofinality of α this problem reduces to characterizing the spaces I^τ, $(I^\tau)^{<\omega}$, $I^{<\tau}$, $I^\tau\times I^{<\omega}$, and $I^\tau\times(I^\kappa)^{<\omega}$ for infinite cardinals $\kappa<\tau$. In fact, such characterizations are known for the first three spaces: I^τ, $(I^\tau)^\infty$ and $I^{<\tau}$.

We distinguish between countable and uncountable cardinals τ. For $\tau=\omega$ the power $I^\tau=I^\omega$ is nothing else but the Hilbert cube. The topological characterization of the Hilbert cube is one of the most brilliant achievements of infinite-dimensional topology and belongs to H.Toruńczyk [To1].

Characterization 2.9 (Toruńczyk)**.** *A topological space X is homeomorphic to the Hilbert cube I^ω if and only if X is a compact metrizable absolute retract satisfying the disjoint cells property in the sense that any two maps $f,g:I^n\to X$ from a finite-dimensional cube can be uniformly approximated by maps with disjoint images.*

A topological characterization of Tychonov cubes I^τ for uncountable cardinals τ is even shorter and belongs to E. Ščepin [S].

Characterization 2.10 (Ščepin)**.** *A topological space X is homeomorphic to a non-metrizable Tychonov cube I^τ if and only if X is a non-metrizable uniform-by-character compact AR of weight τ.*

A topological space X is called *uniform-by-character* if the character at each point of X equals the character of X.

To give a topological characterization of spaces $(I^\tau)^{<\omega}$ and $I^{<\tau}$ we need to recall the notion of a strongly universal space.

Definition 1. Let $\mathcal{K}$ be a class of compact Hausdorff spaces. A topological space X is defined to be

- *universal for the class* $\mathcal{K}$ if each compact subspace of X belongs to $\mathcal{K}$ and each compactum $K\in\mathcal{K}$ is homeomorphic to some compact subset of X;
- *strongly universal for the class* $\mathcal{K}$ if each compact subspace of X belongs to $\mathcal{K}$ and for any compact space $K\in\mathcal{K}$ any embedding $f:B\to X$ of a closed subset B of K can be extended to an embedding $\bar f:K\to X$ of the whole K;
- *strongly universal* if X is strongly universal for some class $\mathcal{K}$ of compacta.

It is easy to see that each strongly universal space X is strongly universal for the class $\mathcal{K}(X)$ of all spaces homeomorphic to compact subsets of X.

We shall say that a topological space X has the *compact unknotting property* if every homeomorphism $h : A \to B$ between compact subsets $A, B \subset X$ extends to an autohomeomorphism of X. It is easy to see that each space with compact unknotting property is strongly universal. The converse is true for k_ω-spaces.

Theorem 2.11 (Unknotting Theorem). *A k_ω-space X is strongly universal if and only if it has the compact unknotting property.*

Another fundamental feature of strongly universal k_ω-spaces is described by

Theorem 2.12 (Uniqueness Theorem). *Two k_ω-spaces X, Y are homeomorphic provided they are strongly universal for some class $\mathcal{K}$ of compact Hausdorff spaces. In particular, two strongly universal k_ω-spaces X, Y are homeomorphic if and only if $\mathcal{K}(X) = \mathcal{K}(Y)$.*

Both the theorems can be proved by the standard back-and-forth argument. In light of the above results it would be helpful to detect ordinals for which the space $I^{<\alpha}$ is strongly universal or has the compact unknotting property.

Theorem 2.13. *For an ordinal α the following conditions are equivalent:*

(1) *$I^{<\alpha}$ is a strongly universal k_ω-space;*
(2) *$I^{<\alpha}$ is a k_ω-space with the compact unknotting property;*
(3) *$\mathrm{cf}(\alpha) = \omega$ and $\beta + |\beta| < \alpha$ for any uncountable ordinal $\beta < \alpha$;*
(4) *$I^{<\alpha}$ is homeomorphic to a topological group;*
(5) *$I^{<\alpha}$ is homeomorphic to a locally convex linear topological lattice.*

Observe that this theorem characterizes ordinals α with countable cofinality for which the space $I^{<\alpha}$ is strongly universal. For ordinals with uncountable cofinality we get another theorem characterizing strongly universal spaces $I^{<\alpha}$.

Theorem 2.14. *For an ordinal α with uncountable cofinality the following conditions are equivalent:*

(1) *$I^{<\alpha}$ is a strongly universal space;*
(2) *$I^{<\alpha}$ has the compact unknotting property;*
(3) *α is a regular cardinal.*

These two theorems imply that for spaces $I^{<\alpha}$ the strong universality is equivalent to the compact unknotting property.

Let us note that for the smallest uncountable ordinal ω_1 the class $\mathcal{K}(I^{<\omega_1})$ of compact subspaces of $I^{<\omega_1}$ is well-understood: it consists of all Corson compacta of weight $\le \omega_1$. We recall that a topological space X is called *Corson compact* if it is homeomorphic to a compact subset of a Σ-product of lines $\Sigma(\mathbb{R}) = \{f \in \mathbb{R}^\tau : |\{i \in \tau : f(i) \neq 0\}| \le \omega\} \subset \mathbb{R}^\tau$ for some cardinal τ.

For an infinite ordinal α with countable cofinality the class $\mathcal{K}(I^{<\alpha})$ also admits a simple description: if $\alpha > \omega$, then $\mathcal{K}(I^{<\alpha})$ consists of all compact Hausdorff spaces with weight $< \alpha$. For the ordinal $\alpha = \omega$ the class $\mathcal{K}(I^{<\omega})$ consists of all finite-dimensional metrizable compact spaces. Using this description and the Uniqueness Theorem we get the following characterization theorems. The first two of them belong to K.Sakai [Sa].

Characterization 2.15 (Sakai). *A topological space X is homeomorphic to the space $I^\infty = I^{<\omega}$ if and only if X is a strongly universal k_ω-space for the class of finite-dimensional compact metrizable spaces.*

Characterization 2.16 (Sakai). *A topological space X is homeomorphic to the space $(I^\omega)^\infty = (I^\omega)^{<\omega}$ if and only if X is homeomorphic to $I^{<\alpha}$ for some countable limit ordinal $\alpha > \omega$ if and only if X is a strongly universal k_ω-space for the class of compact metrizable spaces.*

The latter characterization theorem of Sakai was generalized to spaces $(I^\tau)^\infty = (I^\tau)^{<\omega}$ by T. Banakh [Ba1].

Characterization 2.17 (Banakh)**.** *A topological space X is homeomorphic to the space $(I^\tau)^{<\omega}$ for some infinite cardinal τ if and only if X is a strongly universal k_ω-space for the class of compact spaces of weight $\le \tau$.*

Finally, the topology of the spaces $I^{<\tau}$ for cardinals τ of countable cofinality was characterized by O. Shabat and M. Zarichnyi in [SZ].

Characterization 2.18 (Shabat, Zarichnyi)**.** *A topological space X is homeomorphic to $I^{<\tau}$ for some cardinal with $\mathrm{cf}(\tau) = \omega$ if and only if X is a strongly universal k_ω-space for the class of compact spaces of weight $< \tau$.*

These theorems give us topological characterizations of strongly universal spaces of the form $I^{<\alpha}$ for ordinals α with countable cofinality. Next, we turn to the problem of topological characterization of the spaces $I^\tau \times (I^\kappa)^{<\omega}$ with $\tau > \kappa$. For $\kappa = 1$ this problem was posed in the paper [SZ]. It should be mentioned that unlike the spaces considered in Theorems 2.15–2.18 the spaces $I^\tau \times (I^\kappa)^\infty$ for $\tau > \kappa$ are not strongly universal.

First we recall two notion. Let κ be a cardinal. A closed subset A of a topological space X is called

- a G_κ*-set* in X if $A = \cap\mathcal{U}$ for some family $\mathcal{U}$ of open subsets of X with $|\mathcal{U}| = \kappa$;
- a $Z_{<\kappa}$*-set* in A is for every map $f : I^\kappa \to X$ and a family $\mathfrak{U}$ of open covers of X with $|\mathfrak{U}| < \kappa$ there is a map $g : X \to X \setminus A$ which is $\mathcal{U}$-near to f for every cover $\mathcal{U} \in \mathfrak{U}$.

Observe that for the cardinal $\kappa = \omega$, the notion of a $Z_{<\omega}$-set coincides with the classical notion of a Z-set introduced by Anderson, see [Ch].

Our final theorem gives a characterization of the spaces $I^\tau \times (I^\kappa)^{<\omega}$ and hence answers the mentioned problem from [SZ].

Characterization 2.19. *For a topological space X and infinite cardinals $\tau \ge \kappa$ the following conditions are equivalent:*

1. *X is homeomorphic to $I^\tau \times (I^\kappa)^{<\omega}$;*
2. *X is homeomorphic to $I^{<\alpha}$ for some ordinal with $|\alpha| = \tau$, $\mathrm{cf}(\alpha) = \omega$, and $|\mathrm{tl}(\alpha)| = \kappa$;*
3. *X is a k_ω-space such that each compact subset $K \subset X$ lies as a $Z_{<\kappa}$-set in some compact G_κ-subset $K \subset X$, homeomorphic to the Tychonov cube I^τ.*

3. Proof of Proposition 2.1

Let X be a compact Hausdorff space with a distinguished point $* \in X$ such that $|X| > 1$ and let α be an ordinal.

Fix any cofinal subset $C \subset \alpha$ with $|C| = \mathrm{cf}(\alpha)$. The space $X^{<\alpha}$ is a k-space since its topology is determined by the family of compacta $\{X^\beta : \beta \in C\}$. Since this family has size $\le \mathrm{cf}(\alpha)$, we get $\mathrm{k}(X^{<\alpha}) \le \mathrm{cf}(\alpha)$.

Next we show that $\mathrm{cf}(\alpha) \le \mathrm{kc}(X^{<\alpha})$. Assume conversely that $\mathrm{kc}(X^{<\alpha}) < \mathrm{cf}(\alpha)$ and fix a cover $\mathcal{K}$ of $X^{<\alpha}$ by compacta with $|\mathcal{K}| = \mathrm{kc}(X^{<\alpha})$. Each set $K \in \mathcal{K}$ can be thought as a compact subset of the Tychonov power X^α.

Pick any point $x_0 \in X \setminus \{*\}$ and consider the constant function $f_0 : \alpha \to \{x_0\}$ in X^α. For every $K \in \mathcal{K}$ this function f_0 does not belong to K. Hence there is a finite subset $F_K \subset \alpha$ such that any function $f \in X^\alpha$ with $f|F_K \equiv x_0$ does not belong to K. Now consider the set $F = \bigcup_{K\in\mathcal{K}} F_K$. Since $|F| \le |\mathcal{K}| < \mathrm{cf}(\alpha)$, there is an ordinal $\beta < \alpha$ with $F \subset [0, \beta)$. Then the function $f : \alpha \to X$ defied by $f|[0, \beta) \equiv x_0$ and $f|[\beta, \alpha) = *$ belongs to $X^{<\alpha}$ but not to $\cup\mathcal{K} = X^{<\alpha}$, which is a contradiction.

Hence $\mathrm{cf}(\alpha) \le \mathrm{kc}(X^{<\alpha}) \le \mathrm{k}(X^{<\alpha}) \le \mathrm{cf}(\alpha)$.

It remains to verify that $\mathrm{nw}(X^{<\alpha}) = \mathrm{nw}(X) \times |\alpha|$. For this we will exploit a well-known equality $\mathrm{nw}(X^\kappa) = \mathrm{nw}(X) \times \kappa$ and holding for any cardinal κ and the equality $|\alpha| = \mathrm{cf}(\alpha) \times \sup\{|\beta| : \beta < \alpha\}$ whose (easy) verification is left to the reader.

Let $C \subset \alpha$ be a cofinal subset of size $|C| = \mathrm{cf}(\alpha)$. Since $X^{<\alpha} = \bigcup_{\beta\in C} X^\beta$, we get

$$\mathrm{nw}(X^{<\alpha}) \le \Big|\sum_{\beta\in C} \mathrm{nw}(X^\beta)\Big| = \Big|\sum_{\beta\in C} |\beta| \times \mathrm{nw}(X)\Big| = \mathrm{nw}(X) \times |\alpha|.$$

To prove the reverse inequality $\mathrm{nw}(X^{<\alpha})| \ge \mathrm{nw}(X) \times |\alpha|$, observe that $X^{<\alpha}$ contains a topological copy of X and thus $\mathrm{nw}(X^{<\alpha}) \ge \mathrm{nw}(X)$. On the other hand, X contains a copy of the discrete two-point space $2 = \{0, 1\}$ and hence $X^{<\alpha}$ contains a topological copy of $2^{<\alpha}$. Now it remains to show that $\mathrm{nw}(2^{<\alpha}) \ge |\alpha|$. This is clear if α is a limit cardinal or α is not a cardinal. So assume that $\alpha = \kappa^+$ is a successor cardinal. Then $\mathrm{cf}(\alpha) = \alpha$. Assuming that $\mathrm{nw}(2^{<\alpha}) < \alpha$, fix a network $\mathcal{N}$ of the topology of $2^{<\alpha}$ with $|\mathcal{N}| < \alpha$. For each non-empty element $N \in \mathcal{N}$ choose a point $x_N \in N$. Then set $\{x_N : N \in \mathcal{N}\} \subset 2^{<\alpha}$ has size $< \mathrm{cf}(\alpha)$ and thus lies in 2^β for some $\beta < \alpha$. But then $2^{<\alpha} \setminus 2^\beta$ is a non-empty open subset of $2^{<\alpha}$ containing no non-empty element of $\mathcal{N}$. This contradiction completes the proof. □

4. Proof of Proposition 2.5

Let X be a topological space with a distinguished point $*$ and α, β be two ordinals.

It follows from the definition of the strong topology on $X^{<\alpha\cdot\beta}$ and $(X^\alpha)^{<\beta}$ that these spaces are canonically homeomorphic.

The proof of the topological equivalence of $X^{<\alpha+\beta}$ and $X^\alpha \times X^{<\beta}$ for compact Hausdorff X is more tricky. Since the strong topology of $X^{\alpha+\beta}$ is stronger than the topology of product $X^\alpha \times X^{<\beta}$, it remains to verify that the former topology is also weaker than the latter one.

For this fix any point $(x, y) \in X^\alpha \times X^{<\beta}$ and an open neighborhood U of (x, y) in the strong topology of $X^{<\alpha+\beta}$. Pick a compact neighborhood $K \subset X^\alpha$ of x such that $K \times \{y\} \subset U$ and consider the set $W = \{z \in X^{<\beta} : K \times \{z\} \subset U\}$. It is easy to see that for each ordinal $\gamma < \beta$ the intersection $W \cap X^\gamma$ is open in X^γ. Consequently, W is open in the strong topology of $X^{<\beta}$ and $K \times W \subset U$ is a neighborhood of (x, y) in the topology of product $X^\alpha \times X^{<\beta}$. This implies that the strong topology on $X^{<\alpha+\beta}$ is weaker than the topology of the product $X^\alpha \times X^{<\beta}$. □

5. Proof of the Coincidence Theorem 2.2

We divide the proof into several lemmas. First some notation. For a subset F of an ordinal α let $\mathrm{pr}_F : X^{<\alpha} \to X^F$ denote the natural projection assigning to each function $f \in X^{<\alpha}$ its restriction $f|F$.

Lemma 5.1. *If α is a cardinal with uncountable cofinality and $(X, *)$ is a pointed space with $\chi(X) < \mathrm{cf}(\alpha)$, then the strong and product topologies on $X^{<\alpha}$ coincide.*

Proof. Given a function $f_0 \in X^{<\alpha}$ and an open neighborhood U of f_0 in the strong topology we should find a neighborhood V of f_0 in the product topology such that $V \subset U$.

Find an ordinal $\alpha_0 < \alpha$ such that $f_0 \in X^{\alpha_0}$. For every $\beta \in \alpha$ with $\beta \ge \alpha_0$ the set $U \cap X^\beta$ is a neighborhood of f_0 in X^β (with respect to the product topology). Consequently, we can find a finite subset $F_\beta \subset \beta$ and an open set $W_\beta \subset X^{F_\beta}$ such that $f_0 \in X^\beta \cap \mathrm{pr}^{-1}_{F_\beta}(W_\beta) \subset U$. We can assume that the set F_β has minimal possible size.

For every $n \in \omega$ let $A_n = \{\beta \in [\alpha_0, \alpha) : |F_\beta| = n\}$. Since $[\alpha_0, \alpha) = \bigcup_{n\in\omega} A_n$ and $\mathrm{cf}(\alpha) > \omega$, for some n the set A_n has size $|A_n| = \alpha$.

Let us show that $\cup\{F_\beta : \beta \in A_n\}$ is bounded in α, more precisely, this set is contained in the ordinal $\beta_0 = \min A_n$. Assuming the converse, we would find an ordinal $\beta \in A_n$ with $F_\beta \not\subset \beta_0$. But then for the finite subset $F = F_\beta \cap \beta_0$ of β_0 and the open subset $W = \mathrm{pr}_F(\mathrm{pr}^{-1}_{F_\beta}(W_\beta))$ of X^F we would get

$$U \supset X^{\beta_0} \cap \mathrm{pr}^{-1}_{F_\beta}(W_\beta) = X^{\beta_0} \cap \mathrm{pr}^{-1}_F(W)$$

and $|F| < n = |F_{\beta_0}|$, which contradicts the choice of F_{β_0}.

Thus for any $\beta \in A_n$ the set F_β is an element of the family $[\beta_0]^n$ of n-element subsets of the ordinal β_0. Taking into account that α is a cardinal with $\alpha > \beta_0$ we conclude that $|A_n| = \alpha > |[\beta_0]^n|$. Consequently, for some finite set $F \in [\beta_0]^n$ the set $A = \{\beta \in A_n : F_\beta = F\}$ has size $|A| = \alpha$.

For this finite set F and every $\beta \in A$ there is an open neighborhood $W_\beta \subset X^F$ of $y_0 = \mathrm{pr}_F(f_0)$ such that $X^\beta \cap \mathrm{pr}_F^{-1}(W_\beta) \subset U$. Since $\chi(X^F) = \chi(X) < \mathrm{cf}(\alpha) = \mathrm{cf}(|A|)$, for some neighborhood W of y_0 in X^F the set $B = \{\beta \in A : W_\beta \supset W\}$ has size $|B| = \alpha$ and hence is cofinal in α. Then for each $\beta \in B$ we get $X^\beta \cap \mathrm{pr}_F^{-1}(W) \subset U$. Taking into account that $X^{<\alpha} = \bigcup_{\beta\in B} X^\beta$ we conclude that $\mathrm{pr}_F^{-1}(W) \subset U$ and hence $V = \mathrm{pr}_F^{-1}(W)$ is a neighborhood of f_0 in the product topology such that $V \subset U$. □

The following lemma proves the "if" part of Coincidence Theorem 2.2.

Lemma 5.2. *If X is a pointed compact Hausdorff space, and α is an ordinal whose tail* $\mathrm{tl}(\alpha)$ *is a cardinal with uncountable cofinality* $\mathrm{cf}(\mathrm{tl}(\alpha)) > \chi(X)$*, then the strong topology on $X^{<\alpha}$ coincides with the product topology on $X^{<\alpha}$.*

Proof. Write $\alpha = \beta + \mathrm{tl}(\alpha)$ for some $\beta < \alpha$. By Proposition 2.5, the strong topology on $X^{<\alpha}$ is equivalent to the topology of the product $X^\beta \times X^{<\mathrm{tl}(\alpha)}$. Since $\mathrm{tl}(\alpha)$ is a cardinal with $\mathrm{cf}(\mathrm{tl}(\alpha)) > \chi(X)$, the latter topology coincides with the product topology (according to the preceding lemma). □

Next, we prove the "only if' part of Coincidence Theorem 2.2. We divide the proof into two lemmas.

Lemma 5.3. *If $(X, *)$ is a pointed T_1-space with $|X| > 1$, then for each ordinal α with* $\mathrm{cf}(\alpha) = \omega$ *the strong topology on $X^{<\alpha}$ differs from the product topology.*

Proof. Since $\mathrm{cf}(\alpha) = \omega$, there is an increasing sequence of ordinals $(\alpha_n)_{n\in\omega}$ with $\sup \alpha_n = \alpha$. For every $n \in \omega$ fix any function $f_n : \alpha \to X$ such that $f_n(\beta) \neq *$ iff $\beta = \alpha_n$.

Then the subset $F = \{f_n : n \in \omega\} \subset X^{<\alpha}$ is closed and discrete in the strong topology but has a cluster point with respect to the product topology of $X^{<\alpha}$. This means that the strong and product topologies on $X^{<\alpha}$ are different. □

Lemma 5.4. *Suppose that X is a pointed T_1-space with $|X| > 1$ and α is an ordinal whose tail* $\mathrm{tl}(\alpha)$ *is not a cardinal. Then the strong topology on $X^{<\alpha}$ differs from the product topology.*

Proof. Let $\kappa = |\mathrm{tl}(\alpha)|$. If $\mathrm{cf}(\alpha) = \omega$, then we can apply the previous lemma. So we assume that $\omega < \mathrm{cf}(\alpha) \le \kappa = |\mathrm{tl}(\alpha)| < \mathrm{tl}(\alpha)$. It follows that for each ordinal $\beta < \alpha$ there is an ordinal γ such that the interval $[\beta, \gamma)$ has size κ. Using this observation construct two cofinal subsets $A, C \subset \alpha$ such that $|C| = \mathrm{cf}(\alpha)$ and $|[0, \beta) \cap A| = |[\beta, \gamma) \cap A| = \kappa$ for any ordinals $\beta < \gamma$ in C. Denote by $[A]^{<\omega}$ the family of all finite subsets of A and let $h : C \to [A]^{<\omega}$ be a bijective function.

Let $*$ be the distinguished point of X and $x_0 \in X \setminus \{*\}$ be any other point of X. Given an ordinal $\gamma \in C$ consider the characteristic function $f_\gamma : \alpha \to \{*, x_0\} \subset X$ of the set $[0, \gamma) \cap A \setminus h(\gamma)$. More precisely,

$$f_\gamma(x) = \begin{cases} x_0 & \text{if } x \in A \cap [0, \gamma) \setminus h(\gamma); \\ * & \text{otherwise.} \end{cases}$$

Now consider the subset $F = \{f_\gamma : \gamma \in C\}$ of $X^{<\alpha}$. It is easy to see that the constant function c mapping $[0, \alpha)$ into $*$ belongs to the closure of the set F in the product topology.

We claim that c is not a cluster point of F in the strong topology. For each $\gamma \in C$ let $A_\gamma = [0, \gamma) \cap A \setminus \cup\{h(\beta) : \beta \in C \cap [0, \gamma)\}$. It follows from the choice of A that for any ordinals $\beta < \gamma$ in C the set $A_\gamma \setminus [0, \beta)$ has size κ. Now for each $\gamma \in C$ let $F_\gamma = \{f \in \{*, x_0\}^\gamma : f|A_\gamma \equiv x_0\} \subset X^{<\alpha}$. Next, consider the union $\tilde F = \bigcup_{\gamma\in C} F_\gamma$ of these sets and observe that for each $\gamma \in C$ the intersection $\tilde F \cap X^\gamma$ equals F_γ and is closed in X^γ. This means that the set $\tilde F$ is closed in the strong topology of $X^{<\alpha}$. Since $c \notin \tilde{F}$ and $F \subset \tilde{F}$, the constant function c does not belong to the closure of F in the strong topology. □

6. Proof of Theorem 2.3

The equivalence (1) ⇔ (2) follows from Proposition 2.1.

The proof of the implication (1) ⇒ (3) is standard and left to the reader while (3) ⇒ (4) is trivial. To prove (4) ⇒ (1) assume that $X^{<\alpha}$ is real complete for some ordinal α with uncountable cofinality and some pointed T_1-space $(X,*)$ containing a point $x_1 \in X \setminus \{*\}$. Let $D = \{*, x_1\}$ and observe that the space $D^{<\mathrm{cf}(\alpha)}$, being homeomorphic to a closed subspace of $X^{<\alpha}$ is real complete. By Lemma 5.2 the strong topology on $D^{<\mathrm{cf}(\alpha)}$ coincides with the product topology. On the other hand, it is well-known (and can be easily shown) that the space $D^{<\mathrm{cf}(\alpha)}$ is not real complete in the product topology.

The implication (1) ⇒ (5) is rather standard and follows from the fact that for an ordinal α with countable cofinality, each compact subset of $X^{<\alpha}$ lies in some X^β, $\beta < \alpha$.

To verify the implication (5) ⇒ (1) assume that $\kappa = \mathrm{cf}(\alpha) > \omega$ and $X^{<\alpha}$ is an AE(0) for some pointed T_1-space (X, x_0) containing a point $x_1 \in X \setminus \{x_0\}$. First we construct a closed embedding φ of $2^{<\kappa}$ into $X^{<\alpha}$. Here $2 = \{0,1\}$ is a discrete two-point space with distinguished point 0.

Fix any increasing function $\xi : \kappa \to \alpha$ whose range is cofinal in α and to each function $f \in 2^{<\kappa}$ assign the function $\varphi_f : \alpha \to X^{<\alpha}$ defined by

$$\varphi_f(\beta) = \begin{cases} x_0 & \text{if } \beta \notin \xi(\kappa); \\ x_0 & \text{if } \beta \in \xi(\kappa) \text{ and } f(\xi^{-1}(\beta)) = 0; \\ x_1 & \text{if } \beta \in \xi(\kappa) \text{ and } f(\xi^{-1}(\beta)) = 1. \end{cases}$$

It is easy to verify that the map $\varphi : f \mapsto \varphi_f$ is a closed embedding of $2^{<\kappa}$ into $X^{<\alpha}$. By Lemma 5.2 the strong and product topologies coincide on $2^{<\kappa}$, so we can consider $2^{<\kappa}$ as a subspace of the Cantor cube 2^κ. In this cube consider the compact subset $D = \{f \in 2^\kappa : |f^{-1}(1)| \le 1\} \subset 2^{<\kappa}$. Since $X^{<\alpha}$ is an AE(0), the map $\varphi|D : D \to X^{<\alpha}$ extends to a continuous map $\Phi : 2^\kappa \to X^{<\alpha}$. Then $\Phi(2^\kappa)$ is a dyadic compact subset of $X^{<\alpha}$ with $\Phi(2^\kappa) \setminus X^\beta \neq \emptyset$ for all $\beta < \alpha$. This contradicts the following lemma.

Lemma 6.1. *If X is a pointed T_1-space, then any dyadic compact subset K of $X^{<\alpha}$ lies in some X^β, $\beta < \alpha$.*

Proof. We recall that a compact space is called *dyadic compact* if it is a continuous image of the Cantor discontinuum 2^τ for some cardinal τ. Assume that $K \setminus X^\beta \neq \emptyset$ for all $\beta < \alpha$. Fix an increasing unbounded function $f : \mathrm{cf}(\alpha) \to \alpha$. For each ordinal $\beta \in \mathrm{cf}(\alpha)$ let $U_\beta = K \setminus X^{f(\beta)}$ and fix a point $x_\beta \in U_\beta$. If $\mathrm{cf}(\alpha) = \omega$, then the infinite set $\{x_\beta : \beta \in \mathrm{cf}(\alpha)\}$ is closed and discrete in the compact space K, which is not possible.

If $\mathrm{cf}(\alpha) > \omega$, then $\{U_\beta : \beta < \mathrm{cf}(\alpha)\}$ is an uncountable family of non-empty open subsets of the dyadic compactum K. Since each regular uncountable cardinal is a caliber of any dyadic compactum, see [En], there is a subset $\Lambda \subset \mathrm{cf}(\alpha)$ of size $|\Lambda| = \mathrm{cf}(\alpha)$ such that $\cap_{\beta\in\Lambda} U_\beta$ contains some point t. This point cannot belong to $X^{<\alpha} = \bigcup_{\beta\in\Lambda} X^{f(\beta)}$, which is a contradiction. □

7. Proof of Theorem 2.14

To get Theorem 2.14, we shall prove the implications (2) ⇒ (1) ⇒ (3) ⇒ (2).

(2) ⇒ (1). We need to show that each space X with compact unknotting property is strongly universal. Take any compact subspace $A \subset X$ and let $f : B \to X$ be an embedding of a closed subset $B \subset A$ into X. Since X has the compact unknotting property, the homeomorphism $f : B \to f(B) \subset X$ extends to a homeomorphism $h : X \to X$. Then the restriction $h|A : A \to X$ is the required embedding of A into X, extending the embedding f.

(1) ⇒ (3). Assume that for some ordinal α with $\mathrm{cf}(\alpha) > \omega$ the space $I^{<\alpha}$ is strongly universal. We need to check that $\mathrm{cf}(\alpha) = \alpha$. Assuming the converse we would get that the Tychonov cube $I^{\mathrm{cf}(\alpha)}$ is a subset of $I^{<\alpha}$. Repeating the argument from the proof of the implication (5) ⇒ (1) of Theorem 2.3 we can find an embedding $f : D \to I^{<\alpha}$ of some closed subset $D \subset I^{\mathrm{cf}(\alpha)}$ which cannot be extended to an embedding (even to a continuous map) $\Phi : I^{\mathrm{cf}(\alpha)} \to X^{<\alpha}$. This shows that $X^{<\alpha}$ is not strongly universal.

In [Ba3] it is shown that for any regular cardinal κ the space $I^{<kappa}$ has the compact unknotting property. This yields the implication (3) ⇒ (2).

8. Proof of Classification Theorem 2.7

First we shall prove one lemma which will help us to recover the cardinal $|\mathrm{tl}(\alpha) - \mathrm{cf}(\alpha)|$ from the topological structure of $X^{<\alpha}$. For this we shall introduce two rather exotic cardinal topological invariants.

We recall that a compact Hausdorff space K is called *dyadic compact* if K is a continuous image of the Cantor discontinuum 2^κ for some cardinal κ. For a dyadic compact subset A of a topological space X let

- $\psi(A, X)$ be the smallest size $|\mathcal{U}|$ of a family $\mathcal{U}$ of open subsets of X such that $A = \cap\mathcal{U}$;
- $\psi_D(A, X) = \sup\{\psi(A, K)^+ : K \text{ is dyadic compact in } X \text{ and } A \subset K\}$.

Here as expected κ^+ stands for the smallest cardinal greater than a given cardinal κ.

Now we are able to introduce our exotic cardinal invariants. For a topological space X let

- $\psi_s(X) = \min\{\psi(K, X) : \ K \text{ is dyadic compact in } X\}$;
- $\psi_D(X) = \min\{\psi_D(K, X) : \ K \text{ is dyadic compact in } X\}$.

The following easy lemma (whose proof relies on Lemma 6.1) expresses the values of the cardinal invariants of the space $X^{<\alpha}$ via the properties of the ordinal α.

Lemma 8.1. *Let X be a compact metrizable pointed space with $|X| > 1$. For any ordinal α we get*

(1) $\psi_s(X^{<\alpha}) = |\mathrm{tl}(\alpha)|$;
(2) $\psi_D(X^{<\alpha}) = \sup\{|\beta|^+ : \beta < \mathrm{tl}(\alpha)\}$.

With Lemma 8.1 in our disposition the proof of Classification Theorem 2.4 becomes quite short. The "if" part easily follows from Reduction Theorem 2.4.

To prove the "only if" part, fix some pointed metrizable compact space X containing more than one point and assume that for infinite ordinals α, β the spaces X^α and X^β are homeomorphic. Applying Proposition 2.1 and Lemma 8.1 we conclude that $|\alpha| = |\beta|$, $\mathrm{cf}(\alpha) = \mathrm{cf}(\beta)$ and $|\mathrm{tl}(\alpha)| = |\mathrm{tl}(\beta)|$. It remains to verify that the cardinals $|\mathrm{tl}(\alpha) - \mathrm{cf}(\alpha)|$ and $|\mathrm{tl}(\beta) - \mathrm{cf}(\beta)|$ are equal. Since these cardinals can take one of two values: 0 or $|\mathrm{tl}(\alpha)| = |\mathrm{tl}(\beta)|$, it suffices to verify that $\mathrm{cf}(\alpha) = \mathrm{tl}(\alpha)$ if and only if $\mathrm{cf}(\beta) = \mathrm{tl}(\beta)$.

Assume that $\mathrm{tl}(\alpha) = \mathrm{cf}(\alpha)$. Then $\mathrm{cf}(\alpha) = \sup\{|\gamma|^+ : \gamma < \mathrm{tl}(\alpha)\} = \psi_D(X^{<\alpha}) = \psi_D(X^{<\beta}) = \sup\{|\gamma|^+ : \gamma < \mathrm{tl}(\beta)\}$, which together with $|\mathrm{tl}(\beta)| = \mathrm{cf}(\beta)$ just implies $\mathrm{tl}(\beta) = \mathrm{cf}(\beta)$. □

9. Tychonov cubes and topological groups

In this section we shall prove that topological copies of Tychonov cubes in topological groups must be small in some sense. This result will be applied in the proof of Theorem 2.13 but seems to have an independent value. We start however with an opposite result.

Lemma 9.1. *If a compact subset G of a Tychonov cube I^κ is homeomorphic to a topological group, then $\psi(G, I^\kappa) = \kappa$.*

Proof. Assume conversely that $\lambda = \psi(G, I^\kappa) < \kappa$. Then we can find an index subset $A \subset \kappa$ of size $|A| \le \lambda$ and a closed subset $K \subset I^A$ such that $G = \mathrm{pr}_A^{-1}(K)$ where $\mathrm{pr}_A : I^\kappa \to I^A$ denotes the natural projection. Taking into account that $|A| < \kappa$ we see that G is homeomorphic to the product $I^\kappa \times K$. Observe that $I^\kappa \times K$ is the limit of the inverse spectrum consisting of compacta $I^C \times K$ with $C \subset \kappa$ having size $|C| \le \lambda$. On the other hand, G is the limit of the inverse spectrum consisting of all quotient groups of G having weight $\le \lambda$. Applying Ščepin Spectral Theorem [FC, 3.1.9] we could find a subset $C \subset \kappa$ of size $|C| \le \lambda$ such that the projection $\mathrm{pr} : I^\kappa \times K \to I^C \times K$ is homeomorphic to a group homomorphism. Then the preimage of any point under the projection pr must be homeomorphic to a topological group. On the other hand, such a preimage, being a Tychonov cube I^κ, cannot be homeomorphic to a topological group (Tychonov cubes have the fixed point property while topological groups have not). □

An opposite result also is true.

Proposition 9.2. *Let G be a topological group and K be a compact subset of G, containing the neutral element e of G. If K is homeomorphic to a Tychonov cube, then $\psi(K,G) \ge \psi(K,KK^{-1}) \ge w(K)$.*

Proof. Assume conversely that $\lambda = \psi(K,KK^{-1}) < w(K) = \kappa$. Identify K with the Tychonov cube I^κ so that the neutral element $e \in K$ of G coincides with the constant zero function $\mathbf{0}$, the distinguished point of I^κ. For a subset $A \subset \kappa$ identify the Tychonov cube I^A with the subset $\{f \in I^\kappa : f|\kappa \setminus A \equiv 0\}$ of I^κ. In particular, $I^\emptyset = \{\mathbf{0}\}$.

For every $n \ge 1$ consider the map $f_n : (I^\kappa \times I^\kappa)^n \to G$ defined by

$$f_n\big((x_1,y_1),(x_2,y_2),\dots,(x_n,y_n)\big) = x_1y_1^{-1}x_2y_2^{-1}\cdots x_ny_n^{-1}.$$

Since $\lambda = \psi(K,KK^{-1}) < \kappa$, we get $\psi\big(f_n^{-1}(K), f_n^{-1}(KK^{-1})\big) \le \psi(K,KK^{-1}) \le \lambda$.

Let $A_0 = \kappa$. By induction we shall construct a decreasing sequence $(A_n)_{n\ge1}$ of subsets of κ such that for every $n \ge 1$ we get

(1) $|A_{n-1} \setminus A_n| \le \lambda$;
(2) $(I^{A_n} \times I^{A_n})^n \subset f_n^{-1}(K)$;
(3) $(I^{A_n} \times I^{A_n})^{n+1} \subset f_{n+1}^{-1}(KK^{-1})$.

Assume that for some n the sets $A_1,\dots,A_{n-1}$ satisfying the conditions (1)–(3) have been constructed. Since $(\mathbf{0},\mathbf{0})^n \in f_n^{-1}(K)$ and

$$\psi\big(f_n^{-1}(K),(I^{A_{n-1}} \times I^{A_{n-1}})^n\big) \le \psi\big(f_n^{-1}(K), f_n^{-1}(KK^{-1})\big) \le \lambda,$$

we can find an index subset $A_n \subset A_{n-1}$ such that $|A_{n-1} \setminus A_n| \le \lambda$ and $(I^{A_n} \times I^{A_n})^n \subset f_n^{-1}(K)$. It is clear that this set satisfies the conditions (1), (2). To see that the condition (3) is satisfied too, fix any sequence $((x_i,y_i))_{i\le n+1} \in (I^{A_n} \times I^{A_n})^{n+1}$ and observe that

$$f_{n+1}\big((x_1,y_1),\dots,(x_n,y_n),(x_{n+1},y_{n+1})\big) = f_n\big((x_1,y_1),\dots,(x_n,y_n)\big) \cdot f_1(y_{n+1},x_{n+1})^{-1} \in KK^{-1}$$

which finishes the inductive step.

Finally let $A = \bigcap_{n\ge1} A_n$ and observe that $|\kappa \setminus A| = \lambda$ and $(I^A \times I^A)^n \subset f_n^{-1}(K)$ for all n. It follows that the closed subgroup H of G generated by $I^A \subset K$ lies in K and thus is compact. Also $\psi(I^A,H) \le \psi(I^A,K) = \psi(I^A,I^\kappa) = \lambda < \kappa = |A|$.

Now it is easy to construct a closed subgroup $N \subset H$ with $\psi(N,H) \le \psi(I^A,H) < |A|$ and $N \subset I^A$. Since $\psi(N,I^A) \le \psi(N,H) < |A|$, we get a contradiction with Lemma 9.1. □

10. Proof of Theorem 2.13

Given an ordinal α we shall prove the implications (1) ⇔ (2) ⇔ (3) ⇒ (5) ⇒ (4) ⇒ (3).

The equivalence (1) ⇔ (2) follows from Theorem 2.11.

(1) ⇒ (3) Assume that $I^{<\alpha}$ is a strongly universal k_ω-space. Theorem 2.3 implies that either α is a successor ordinal (and then $I^{<\alpha}$ is a Tychonov cube) or $\mathrm{cf}(\alpha) = \omega$. The first case is not possible since Tychonov cubes are not strongly universal. Thus $\mathrm{cf}(\alpha) = \omega$.

Now we show that for any uncountable ordinal $\beta < \alpha$ we get $\beta + |\beta| < \alpha$. Observe that $I^{\beta+|\beta|}$ is homeomorphic to the compact subset I^β of $X^{<\alpha}$. Then the strong universality of $X^{<\alpha}$ allows us to extend the identity embedding of I^β in $I^{<\alpha}$ to an embedding $e : I^{\beta+|\beta|} \to I^{<\alpha}$. The range

$e(I^{\beta+|\beta|})$, being a compact subset of $I^{<\alpha}$, lies in the Tychonov cube I^γ for some $\gamma \in [\beta, \alpha)$ (to prove this fact, apply the equality $\mathrm{cf}(\alpha) = \omega$). Write $\gamma = \beta + \delta$ and observe that

$$|\beta| = \psi(I^\beta, I^{\beta+|\beta|}) = \psi\big(I^\beta, e(I^{\beta+|\beta|})\big) \le \psi(I^\beta, I^\gamma) = \psi(I^\beta, I^{\beta+\delta}) = \max\{\omega, |\delta|\}$$

which implies the desired inequality

$$\beta + |\beta| \le \alpha + |\delta| \le \alpha + \delta = \gamma < \alpha$$

completing the proof of the implication (1) ⇒ (3).

To prove that (3) ⇒ (1), assume that $\mathrm{cf}(\alpha) = \omega$ and $\beta + |\beta| < \alpha$ for any uncountable ordinal $\beta < \alpha$. According to Theorem 2.3, the equality $\mathrm{cf}(\alpha) = \omega$ implies that $I^{<\alpha}$ is a k_ω-space. To show that it is strongly universal, fix any compact subset $K \subset I^{<\alpha}$ and an embedding $f : B \to X^{<\alpha}$ of a closed subset $B \subset K$. Using the compactness of $f(B)$ and K in $X^{<\alpha}$, find an ordinal $\beta < \alpha$ such that $K \cup f(B) \subset I^\beta$. Since I^β is an AE, the map f admits a continuous extension $\Phi : K \to I^\beta$.

To continue the proof consider separately three cases:

1. $\alpha = \omega$. In this case the $\beta < \omega$ and thus $K \subset I^\beta$ is a metrizable finite dimensional compactum and so is the quotient space K/B. Consequently, for some $m < \omega$ there is an embedding $e : K/B \to I^m$ such that $e(\{B\})$ is the distinguished point of I^m. Denote by $\pi : K \to K/B$ the quotient map. Then the map $\bar f : K \to I^\beta \times I^m$ defined by $\bar f(x) = (\Phi(x), e \circ \pi(x))$, $x \in K$, is an embedding of K into $I^{\beta+m} \subset I^{<\omega}$ extending the embedding f. This completes the proof of the strong universality of $I^{<\alpha}$ in the case $\alpha = \omega$.

2. $\alpha > \omega$ is a countable ordinal. Since $\mathrm{cf}(\alpha) = \omega$ this ordinal is limit and hence $\beta + 1 < \alpha$. Replacing the ordinal β by a larger ordinal, if necessary, we can assume that β is infinite. In this case $I^\beta \times [0, 1]$ is a Hilbert cube containing I^β as a Z-set. It follows that $f : B \to I^\beta \times [0, 1]$ is a Z-embedding (the latter means that the range $f(B)$ is a Z-set in $I^\beta \times [0, 1]$). One of the basic results of the theory of Hilbert cube manifolds asserts that each Z-embedding of a closed subset of a metrizable compactum into the Hilbert cube extends to a Z-embedding of the whole compactum, see [Ch, 11.2]. Consequently the Z-embedding $f : B \to I^\beta \times [0, 1]$ of B can be extended to an embedding of K into $I^\beta \times [0, 1] \subset I^{\beta+1} \subset I^{<\alpha}$. This completes the proof of the strong universality of $X^{<\alpha}$ in case of countable ordinal $\alpha > \omega$.

3. The ordinal α is uncountable. In this case our assumption yields $\beta + |\beta| < \alpha$. Without loss of generality we can assume that the ordinal β is infinite. The compactum K, being a subspace of I^β, has weight $w(K) \le |\beta|$. Then the quotient space K/B has weight $w(K/B) \le |\beta|$ too and hence admits an embedding $e : K/B \to I^{|\beta|}$ such that $e(\{B\})$ is the distinguished point of $I^{|\beta|}$. Let $\pi : K \to K/B$ denote the quotient map and observe that the formula $\bar f(x) = (\Phi(x), e \circ \pi(x))$, $x \in K$, defines an embedding $\bar f : K \to I^{\beta+|\beta|} \subset I^{<\alpha}$ of K that extends the embedding $f : B \to X^{<\alpha}$.

Next, we prove the implication (3) ⇒ (5). Assume that $\mathrm{cf}(\alpha) = \omega$ and $\beta + |\beta| < \alpha$ for each uncountable $\beta < \alpha$. We shall construct a locally convex linear topological lattice homeomorphic to $I^{<\alpha}$.

Fix an increasing sequence $(\alpha_n)_{n\in\omega}$ of ordinals with $\alpha = \sup\{\alpha_n : n \in \omega\}$ and for every $n \in \omega$ let $L_n = \{f \in \mathbb{R}^\alpha : f([0, \alpha_n)) \subset [-2^n, 2^n]$ and $f([\alpha_n, \alpha)) = \{0\}\}$. Consider the union $L = \bigcup_{n\in\omega} L_n$ and note that L is a linear sublattice of the vector lattice $\mathbb{R}^\alpha$ endowed with the coordinatewise operations of minimum and maximum. Endow L with the strongest topology inducing the Tychonov product topology on each compactum L_n. By a standard argument show that L is a locally convex linear topological lattice with respect to this strong topology.

By analogy with the proof of the implication (3) ⇒ (1) it can be shown that the space L is strongly universal for the class of compact subspaces of $I^{<\alpha}$. Then Uniqueness Theorem 2.12 implies that $I^{<\alpha}$ is homeomorphic to the locally convex linear topological lattice L.

The implication (5) ⇒ (4) is trivial. To prove that (4) ⇒ (3), assume that $I^{<\alpha}$ admits a compatible group operation which can be chosen so that the distinguished element of $I^{<\alpha}$ is the neutral element of the group.

First we show that $\mathrm{cf}(\alpha) = \omega$. Indeed, α cannot be a successor ordinal, since Tychonov cubes are not homeomorphic to topological groups (because of the fixed point property). Assuming that $\mathrm{cf}(\alpha) > \omega$ we would get that $I^{<\alpha}$ is a countably compact group whose Raikov completion is a compact topological group, homeomorphic to $\beta(I^{<\alpha}) = I^\alpha$ (see [Com, 6.5]), which is not possible.

Thus $\mathrm{cf}(\alpha) = \omega$. It remains to verify that $\beta + |\beta| < \alpha$ for any uncountable ordinal $\beta < \alpha$. Given an uncountable ordinal $\beta < \alpha$, consider the compact subset I^β of the topological group $I^{<\alpha}$. Replacing β by a larger ordinal, if necessary, we can assume that the cube I^β contains the neutral element of the group $I^{<\alpha}$. The group product $I^\beta \cdot (I^\beta)^{-1}$, being a compact subset of $I^{<\alpha}$, lies in the Tychonov cube I^γ for some ordinal $\gamma < \alpha$. Write $\gamma = \beta + \delta$. By Proposition 9.2,

$$|\beta| = w(I^\beta) \le \psi(I^\beta, I^\beta \cdot (I^\beta)^{-1}) \le \psi(I^\beta, I^\gamma) = \max\{\omega, |\delta|\} = |\delta|$$

and hence $\beta + |\beta| \le \beta + |\delta| \le \beta + \delta = \gamma < \alpha$.

11. Proof of Characterization 2.19

Lemma 11.1. *Let κ be an infinite cardinal and let Z be a G_κ-subset of a Tychonov cube $X = I^\tau$ with $\tau > \kappa$. If Z is a $Z_{<\kappa}$-set in X and Z is an absolute retract, then the pair (X, Z) is homeomorphic to the pair $(I^\tau \times I^\kappa, I^\tau \times \{0\}^\kappa)$.*

Proof. Since Z is a closed G_κ-set in I^τ, there is a countable subset $K \subset \tau$ of size $|K| = \kappa$ and a closed subset $F \subset I^K$ such that $Z = \mathrm{pr}^{-1}(F)$ where $\mathrm{pr} : I^\tau \to I^K$ denotes the natural projection. Thus the pair (X, Z) is homeomorphic to the pair $(I^K \times I^A, F \times I^A)$, where $A = \tau \setminus K$. It follows from the $Z_{<\kappa}$-property of the set Z in X that the set F is a $Z_{<\kappa}$-set in the Tychonov cube I^κ. Moreover, the space K being homeomorphic to a retract of Z is an absolute retract. Write the uncountable set A as the disjoint union $A = B \cup C$, where the set B has size $|B| = \kappa$. By Edwards' ANR-Theorem [Ch, 44.1] or Ščepin characterization theorem for Tychonov cubes [S], the product $F \times I^B$ is homeomorphic to the Tychonov cube I^κ. Thus we obtain a pair $(I^\tau, Z) = (I^K \times I^B, F \times I^B)$ of two Tychonov cubes with $F \times I^B$ being a $Z_{<\kappa}$-set in $I^K \times I^B$. By $Z_{<\kappa}$-Unknotting Theorem for Tychonov cubes [Chi, ??], this pair is homeomorphic to the pair $(I^\kappa \times I^\kappa, I^\kappa \times \{0\}^\kappa)$. Then the pair $(X, Z) = (I^K \times I^B \times I^D, F \times I^B \times I^C)$ is homeomorphic to $(I^\kappa \times I^\kappa \times I^C, I^\kappa \times \{0\}^\kappa \times I^C) \approx (I^\tau \times I^\kappa, I^\tau \times \{0\}^\kappa)$. □

Now we are ready to prove the characterization Theorem 2.19. The implications (1) ⇒ (2) ⇒ (3) are evident. To prove that (3) ⇒ (1) let $X = \varinjlim X_i$, where (X_i) is an increasing sequence of compact subspaces of X. By condition 3, X_1 is a subset of a compact G_δ-subset $Y_1 \subset X$ homeomorphic to the Tychonov cube I^τ. Proceeding by induction, we construct an increasing sequence of compact G_κ-subsets $Y_i \subset X$ such that $Y_i \supset Y_{i-1} \cup X_i$ and Y_i is homeomorphic to the Tychonov cube I^τ. Then, obviously, $X = \varinjlim Y_i$.

Define an increasing sequence (Z_i) of compact subspaces of X as follows. Let $Z_1 = Y_1$. Suppose that Z_j are already defined for all $j < i$. There exists a compact G_κ-subset $Z_i \subset X$ such that $Z_{i-1} \cup Y_i$ is a $Z_{<\kappa}$-set in Z_i and Z_i is homeomorphic to the Tychonov cube I^τ. Then $X = \varinjlim Z_i$ and every Z_i is a G_κ-subset of Z_{i+1} and also a $Z_{<\kappa}$-set in Z_{i+1}. By Lemma 11.1, $X = \varinjlim Z_i$ is homeomorphic to the direct limit of the sequence

$$I^\tau \to I^\tau \times \{0\}^\kappa \hookrightarrow I^\tau \times I^\kappa \to I^\tau \times I^\kappa \times \{0\}^\kappa \hookrightarrow I^\tau \times I^\kappa \times I^\kappa \to \dots,$$

which is homeomorphic to $I^\tau \times (I^\kappa)^{<\omega}$.

References

[Ba1] T. Banakh. Parametric results for certain classes of infinite-dimensional manifolds (in Russian) // Ukr. Math. Zhurn. **43** (1991), 853–859.

[Ba2] T. Banakh, *On topological groups containing a Fréchet-Urysohn fan*, Matem. Studii **9**:2 (1998), 149–154.

[Ba3] T. Banakh and ??, Characterizing the topology of Σ-products of intervals, in preparation.

[Ch] T.A. Chapman. *Lectures on Hilbert cube manifolds*, CBMS **28**, Providence, 1975.

[Chi] A.Chigogidze, Inverse spectra, North-Holland Math. Library, Vol.53, North-Holland, Amsterdam, 1996.

[Com] W.W. Comfort, *Topological Groups*, in: K.Kunen, J.E.Vaughan (eds.), *Handbook of Set-Theoretic Topology* (North-Holland, Amsterdam, 1984), 1143–1263.

[vD] E.K. van Douwen, *The integers and Topology*, in: K.Kunen, J.E.Vaughan (eds.), *Handbook of Set-Theoretic Topology* (North-Holland, Amsterdam, 1984), 111–167.

[En] R. Engelking, *General Topology* (Warszawa, PWN, 1977).

[FC] V.V. Fedorchuk, A.Ch. Chigogidze, *Absolute retracts and infinite-dimensional manifolds*, Nauka, Moscow, 1992 (in Russian).

[FST] S.P. Franklin, B.V. Smith Thomas, *A survey of k_ω-spaces*, Topology Proc. **2** (1977), 111–124.

[Mi] J. van Mill. Infinite-Dimensional Topology. Prerequisites and introduction, North-Holland, 1989.

[Pe] E. Pentsak, On manifolds modeled on direct limits of $\mathcal{C}$-universal ANR's, Matematychni Studii. 5 (1995), 107–116.

[Sa] K. Sakai. *On $\mathbb{R}^\infty$-manifolds and Q^∞-manifolds*, Topology Appl. **18** (1984), 69–79.

[SZ] O.Shabat, M.Zarichnyi. *Universal maps of k_ω-spaces*, Matem. Studii, ???.

[S] E.Ščepin, *On Tychonov manifolds* // DAN SSSR. 1979. T.246, N3. P.551–554.

[To1] H.Toruńczyk. *On CE-images of the Hilbert cube and characterization of Q-manifolds* // Fund. Math. 1980. V.106, P.31–40.

[To2] H. Toruńczyk. *Characterizing Hilbert space topology*, Fund. Math. **111** (1981), 247–262.

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