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COMPACT SEMILATTICES WITH OPEN PRINCIPAL FILTERS

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Abstract

A locally compact semilattice with open principal filters is a zero-dimensional scattered space. Cardinal invariants of locally compact and compact semilattices with open principal filters are investigated. Structure of topological semilattices on the one-point Alexandroff compactification of an uncountable discrete space and linearly ordered compact semilattices with open principal filters are researched.

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0. Introduction

This paper is a continuation of the work of the first author (see [8, 6]).

A *topological inverse semigroup* is an inverse semigroup defined on a Hausdorff topological space such that the multiplication is jointly continuous and the inversion is continuous.

We follow the terminology of [1, 3, 4, 9, 10]. Let S be a topological inverse semigroup and E the band of S. We define the maps $\varphi\colon S \to E$ and $\psi\colon S \to E$ by the formulae $\varphi(x) = xx^{-1}$ and $\psi(x) = x^{-1}x$.

By Ω we denote the class of all ordinal numbers. Put $\Omega(\alpha) = \{\beta \in \Omega \mid \beta \leq \alpha\}$ for all $\alpha \in \Omega$. The set $\Omega(\alpha)$ is well-ordered by the natural order $\leq$, that is, $\gamma \leq \beta$ if $\Omega(\gamma) \subseteq \Omega(\beta)$ for each $\gamma, \beta \in \Omega(\alpha)$. By ω we denote the first infinite ordinal and by ω_1 we denote the first uncountable ordinal. Further, we identify all cardinals with their corresponding initial ordinals. The successor cardinal of λ is denoted by λ^+.

By $|X|$, $w(x)$, $d(X)$, $\chi(X)$, $c(X)$, $t(X)$, $\pi\chi(X)$ we denote cardinality, weight, density, character, cellularity, tightness, and π-character of a topological space X, respectively.

A *band* is an idempotent semigroup and a *semilattice* is a commutative band. Let E be a semilattice. For $e, f \in E$, we write $e \le f$ if $ef = fe = e$. This defines a partial order on E which we call the *natural order on* E. An idempotent $e \in E$ is called *maximal* (*minimal*) if $ef \neq e$ ($ef \neq f$) for all $f \in E \setminus \{e\}$. Further, by $\operatorname{Max} E$ we denote *the set of all maximal idempotents of* E. We mean the natural partial order in E when we use an order relation in E like $\le$, $<$ unless otherwise stated. If $e \in E$, we write ${\downarrow}e = \{f \in E \mid ef = fe = f\}$, ${\uparrow}e = \{f \in E \mid ef = fe = e\}$ and $NO_E(e) = E \setminus ({\downarrow}e \cup {\uparrow}e)$. If $A \subseteq E$, we put ${\uparrow}A = \bigcup\{{\uparrow}e \mid e \in A\}$, ${\downarrow}A = \bigcup\{{\downarrow}e \mid e \in A\}$.

DEFINITION. A topological semilattice E is called a *semilattice with open principal filters* if the set ${\uparrow}e$ is open in S for each $e \in E$.

1. Properties of compact semilattices with open principal filters

An element e of a topological semilattice E is called *a local minimum* if there exists an open neighbourhood $U(e)$ of e such that ${\downarrow}e \cap U(e) \subseteq {\uparrow}e$ [9]. If E is a topological semilattice with open principal filters, then the set of all local minima of E coincides with E. The *set of all local minima of* E will be denoted by $K(E)$. An element e in E is called the *maximum* or *identity* of E if $ef = fe = f$ for all f in E. We call the element e the *minimum* or *zero* of E if $ef = fe = e$ for all f in E.

PROPOSITION 1.1. *Let E be a topological semilattice and U be an open subset of E. Then ${\uparrow}U$ is open subset in E.*

THEOREM 1.2. *Topological semilattice E is a semilattice with open principal filters if and only if $K(E) = E$.*

This follows easily from the definition of $K(E)$.

REMARK. If E is a topological semilattice with open principal filters, then ${\uparrow}e$ is an open and closed subsemilattice of E for all $e \in E$. If E is a topological semilattice and $e \in E$, then ${\downarrow}e$ and ${\uparrow}e$ are closed in E.

LEMMA 1.3. *Let E be a locally compact semilattice with open principal filters. Then E is zero-dimensional.*

PROOF. Let V be a nonempty connected subset of E. Let $e \in V$. Since ${\uparrow}e$ is open and closed and $e \in V$, we have that ${\uparrow}e \supseteq V$. Hence $fe = ef = e$ for all

$f \in V$. So it follows that if $e, f \in V$, then $ef = fe = f$ also. Hence $e = f$. Thus we have that E is totally disconnected. Since E is locally compact we see that E is zero-dimensional. □

DEFINITION ([11]). A topological semilattice which has a basis of subsemilattices is called a *Lawson semilattice*.

PROPOSITION 1.4. *A locally compact semilattice with open principal filters is a Lawson semilattice.*

Proposition 1.4 follows easily from [11, Theorem 2.1].

REMARK. Let E be a locally compact semilattice with open principal filters. Then the collection of all open subsemilattices of E form a basis for the topology of E.

DEFINITION ([13]). A topological space X is called *scattered* if every nonempty subset A of X contains a point p which is isolated in A.

We recall that it is shown in [13] that a topological space is scattered if and only if every closed subset has an isolated point with respect to that subset.

DEFINITION. Let X be a semilattice. Let $A \subseteq E$. A *minimal element* of A is an element e in A so that if $f \in A$ and $ef = fe = e$, then $e = f$. An element e in A is called the *least* in A (also called a *zero* or a *minimum* of A) if $ef = fe = e$ for all f in A. Similarly we define a *maximal element* of A and a *largest element* of A (also called a *maximum* of A). A *well ordered sequence* in the semilattice X is a function from a well ordered set J into X. It is denoted as (x_α) or $(x_\alpha)_{\alpha \in J}$. If the order in J is denoted as $\leq$, then the well ordered sequence (x_α) is said to be *well ordered increasing* (*decreasing*) *under the natural order* if whenever $\alpha, \beta \in J$ and $\alpha \leq \beta$ we have $x_\alpha x_\beta = x_\alpha$ ($x_\alpha x_\beta = x_\beta$). A *well ordered increasing sequence* in A means a well ordered sequence in A which is increasing under the natural order.

LEMMA 1.5. *Let E be a topological semilattice and $A \subseteq E$ compact. Then A contains a minimal element in A and a maximal element in A. Furthermore, if A is a subsemilattice, then there is a minimum of A.*

PROOF. We prove the existence of a maximal element of A. The proof of the existence of a minimal element in A is similar. Consider the collection of all well ordered increasing sequences in A. It is easily seen that there is such a well ordered sequence F which is maximal under extension. For $e \in F$, put $K(e) = A \cap (\uparrow e)$. Then the collection $\{K(e) \mid e \in F\}$ is a collection of nonempty compact sets which is a chain under containment relation. So $\bigcap\{K(e) \mid e \in F\}$ is not empty. Let

$f \in \bigcap\{K(e) \mid e \in F\}$. Then f should belong to the range of F, since otherwise we could get a larger well ordered increasing sequence in A, by adding f to F, contradicting the maximality of F. It is clear that f is a maximal element of A. If $g \in F$ and $gf = f$ and $f \neq g$, then we get a larger well ordered sequence in A by adding g to F which would contradict the maximality of F. Now suppose that A is a subsemilattice as well. Let e be a minimal element of A. If $f \in A$, then $ef \in A$ and $ef \leq e$. So $ef = e$. So e is the minimum of A. □

THEOREM 1.6. *Let E be a locally compact semilattice with open principal filters. Then E is scattered.*

PROOF. Let A be a closed nonempty subset of E. By Lemma 1.3 there is a subset F of A which is open and closed in A. By Lemma 1.5 there is a maximal element f of F. Since $\{f\} = (\uparrow f) \cup F$ we see that f is isolated in F and hence in A. So E is scattered by [13]. □

If E is a topological space we denote *the set of all its isolated points* by $Is(E)$.

THEOREM 1.7. *Let E be a locally compact semilattice with open principal filters. Then the following hold*:

(i) *$Is(E)$ is dense in E.*
(ii) $w(E) = |E|$.
(iii) $c(E) = d(E) = |Is(E)|$.
(iv) *If in addition E compact, then $\chi(E) = |E|$.*

PROOF. Suppose E is scattered. So (i), (ii) and (iii) follows from [13]. Now (iv) follows from [2, Theorem I.25]. □

REMARK. If E is taken to be only locally compact in Theorem 1.7, then it does not follows that $|E| = \chi(E)$. As an example take a discrete uncountable semilattice.

The following example shows that for every cardinal λ there is a compact semilattice with open principal filters and whose cardinality is λ.

EXAMPLE 1. Let α be an ordinal. Put

$$\mathscr{B} = \{(x, y] = \{z \in \Omega(\alpha) \mid y < z \leq x\} \mid x, y \in \Omega(\alpha)\ \&\ y < x\} \cup \{0\},$$

where 0 is the order type of the empty set. Let τ_Ω be the topology with base $\mathscr{B}$ on $\Omega(\alpha)$. Define a multiplication '$*$' on $\Omega(\alpha)$ by: $\beta * \gamma = \max\{\beta, \gamma\}$ for all $\beta, \gamma \in \Omega(\alpha)$. Then $(\Omega(\alpha), *, \tau_\Omega)$ is a topological semilattice with open principal filters of cardinality $|\alpha|$.

DEFINITION. A topological semilattice E is called an α^-*-semilattice* if E is topologically isomorphic to $(\Omega(\beta), *, \tau_\Omega)$ for some ordinal β.

EXAMPLE 2. Let $\mathscr{A}(\tau)$ be the one-point Alexandroff compactification of a discrete space of cardinality τ with ∞ as its point at infinity. Put $xy = \infty$ if $x \neq y$ and $xx = x$ for all $x, y \in \mathscr{A}(\tau)$. Then $\mathscr{A}(\tau)$ is a compact topological semilattice with open principal filters and cardinality τ.

Example 1 and Example 2 show that there are compact semilattices E_1 and E_2 with open principal filters such that $t(E_1) = |E_1|$ and $t(E_2) < |E_2|$.

The following three questions arise naturaly:

(1) Does every compact scattered space E admit a structure of a topological semilatice with open principal filters?
(2) For every compact space X, the inequality $\pi\chi(X) \leq t(X)$ [19] holds. Is there a compact semilatice E with open principal filters so that $\pi\chi(E) < t(E)$?
(3) Is there a compact semilatice E with open principal filters so that $d(E) < \chi(E)$?

Question (1) can be answered in the negative under the set theoretic assumption that $\beta\mathbb{N} \setminus \mathbb{N}$ has p-points. Let us recall that a point p of a topological space E is called a *p-point* if any countable intersection of neighbourhoods of p is a neighbourhood of p. There are set theoretic models in which $\beta\mathbb{N} \setminus \mathbb{N}$ has no p-points. Continuum Hypothesis (CH) and Martin's Axiom (MA) imply the existence of a p-point in $\beta\mathbb{N} \setminus \mathbb{N}$.

DEFINITION ([5, 14, 12]). A *Franklin-Rajagopalan space* is a compact scattered space X with a countable dense set D of isolated points so that the subspace $X \setminus D$ is homeomorphic to the ordinal space $[1, \omega_1]$ with its usual topology $(\Omega(\omega_1), *, \tau_\Omega)$ of Example 1.

REMARK. The methods of [5] show that in every model of set theory where $\beta\mathbb{N} \setminus \mathbb{N}$ has p-point there are Franklin-Rajagopalan spaces with the additional property that no sequence of isolated points converges to ω_1. We denote by $\gamma\mathbb{N}$ one such space.

EXAMPLE 3. The Franklin-Rajagopalan space $\gamma\mathbb{N}$ (see [5]) is a compact scattered space which does not admit the structure of a topological semilattice with open principal filters.

PROOF. Suppose that $\gamma\mathbb{N}$ admits a structure of topological semilattice with open principal filters. Then $\uparrow\omega_1$ is a compact open semilattice. Put $Y = \uparrow\omega_1$. Let D be the set of all isolated points of Y. Let $M = Y \setminus (D \cup \{\omega_1\})$. Now there is an element γ in $[1, \omega_1]$ so that $[\gamma, \omega_1] \subseteq Y$. We claim that there is an element c in M so that $c > \gamma$ and if x, y are in M and $x > c$ and $y > c$ in the usual order of ω_1 then $xy \neq \omega_1$. For suppose that there is no such element c. Then there are x_1

and y_1 in M so that $y_1 > x_1 > \gamma$ and $x_1 y_1 = \omega_1$. Then there are x_2 and y_2 in M so that $\gamma < x_1 < y_1 < x_2 < y_2$ and $x_2 y_2 = \omega_1$. By induction we have a sequence $x_1 < y_1 < x_2 < y_2 < \cdots < x_n < y_n < \cdots$ in M so that $x_n y_n = \omega_1$ for all $n = 1, 2, \ldots$. Then there is an element α in M so that $\operatorname{lt} x_n = \operatorname{lt} y_n = \alpha$. Then by continuity of multiplication we get $\alpha = \omega_1$ which is a contradiction. So there is an element c in M with $c > \gamma$ so that if x, y are in M and $x, y > c$, then $xy \neq \omega_1$. It is also easy to see, by using continuity of multiplication, that if x, y are in Y and both x, y are $\neq \omega_1$, then there can be at most a countable number of elements a in Y with $ax = y$. So there is an element $p \in M$ such that $p > c$ and if $q \in M$ and $q > p$ then qx does not belong to D for every x in D. Now put $S = Y \cap [1, p + 1]$. Then S is a compact subset of Y and does not contain ω_1. So there is a compact open set W in Y such that $\omega_1 \in W$ and if $x \in W \cap M$, then $x > p$ in the natural order of $[1, \omega_1]$. Let $q \in W \cap M$. Let $B = \{k \mid k \in D \text{ and } kq = \omega_1\}$. Then ω_1 cannot belong to the closure of B. For suppose that ω_1 is in the closure of B in Y. Since no sequence of isolated points in the space $\gamma\mathbb{N}$ converges to ω_1, it follows that the closure of B in Y is uncountable. So there is a point r in the closure of B such that $r \in M$ and $r > p$ in the order of $[1, \omega_1]$. Then r is a limit of a sequence $s_1, s_2, s_3, \ldots, s_n, \ldots$ from B. By continuity of multiplication we have $qr = \omega_1$. But $q > p > c$ and $r > p > c$ and q, r are in M. So $qr \neq \omega_1$. This contradiction shows that ω_1 does not belong to the closure of B. Let $C = D \setminus B$. Then if y in C, then $qy \neq \omega_1$ and since qy is not in D we see that $qy \in M$, and hence the set $qC = \{qv \mid v \in C\}$ is a countable subset of M. So ω_1 is not in the closure of qC. But ω_1 is in the closure of D and not in the closure of B. So ω_1 is in the closure of C and hence in the closure of qC by continuity of multiplication. This contradiction shows that $\gamma\mathbb{N}$ is a compact scattered space that does not admit the structure of topological semilattice with open principal filters. $\square$

EXAMPLE 4. Let X be the quotient space of the space $[1, \omega_1]$ with topology as in Example 1, where we identify the points ω and ω_1. We define multiplication as follows in X. We put $xy = yx = \max\{x, y\}$ for all $x, y \in X$ if either both $x, y > \omega$ or both x, y are $< \omega$. If one of $x, y < \omega$ and the other $> \omega$ then put $xy = yx = \omega_1$. We also put $\omega_1 x = x\omega_1 = \omega_1$ and $xx = x$ for all $x \in X$. Then X is a compact semilattice with open principal filters. Clearly, $\pi\chi(X) = \omega$ and $t(X) = \omega_1$. So we have a compact scattered space which is also a semilattice with open principal filters and $\pi\chi(X) < t(X)$. This solves Question (2) above.

2. Some classes of compact semilattices with open principal filters

A semilattice E is called *linearly ordered* (*well-ordered*) if the multiplication induces on E a linear order (a well-order).

Let E be a linearly ordered semilattice, and $\leq$ be a natural order on E. By $\leq^d$ we denote *a dual order on* E, that is, $e \leq^d f$ if and only if $ef = f$, for all $e, f \in E$. Obviously, if E is a linearly ordered semilattice, then $\leq^d$ is a linear order on E.

LEMMA 2.1. *Let E be a linearly ordered compact commutative band with open principal filters. Then $\leq^d$ is a well-order.*

PROOF. Let A be any non-empty subset of E. We shall prove that $\inf_{\leq^d} A \in A$.

If there exists a compact subset K in E such that $A \subseteq K$, then the family $\{\uparrow a \cap K \mid a \in A\}$ is centered and $\inf_{\leq^d} A \in \bigcap\{\uparrow a \cap K \mid a \in A\} \subseteq K$.

Put $a = \inf_{\leq^d} A$. If $a \in A$, then the proof is complete. In the other case, $\uparrow a \cap A = \varnothing$ and the set $\uparrow a$ is clopen in E. Thus the set $E \setminus \uparrow a$ is compact and $A \subseteq E \setminus \uparrow a$. Then $\inf_{\leq^d} A \in E \setminus \uparrow a$, but $a \notin E \setminus \uparrow a$, a contradiction. Therefore, $\uparrow a \cap A \neq \varnothing$ and $a \in A$. □

PROPOSITION 2.2. *Every well-ordered semilattice E is algebraicly isomorphic to a subsemilattice of* $(\Omega(\alpha), \min)$ *for some $\alpha \in \Omega$.*

PROOF. Since the cardinality of E is bounded, by [1, Theorem 3.11′] the well-ordered set E is similar to some interval of $\Omega(\alpha)$ (where $\alpha \geq |E|^+$). We denote this similar map by f. Obviously, f is an algebraic isomorphism of E into $(\Omega(\alpha), \min)$. □

THEOREM 2.3. *Every linearly ordered compact semilattice E with open principal filters is an α^--semilattice.*

PROOF. By Lemma 2.1, $\leq^d$ is a well-order on E and by Proposition 2.2 there exists an algebraic isomorphism $f : E \to \Omega(\delta)$ (for some $\delta \leq |E|^+$). Obviously, E has a zero 0 and we put $f(0) = \beta \in \Omega(\delta)$. It is easy to see that $(f(E), \max)$ is an α^--semilattice and the isomorphism $f : E \to \Omega(\delta)$ is continuous. □

$\mathscr{A}(\tau)$ is *the one-point Alexandroff compactification of the discrete space X of cardinality* τ, and $\{a\} = \mathscr{A}(\tau) \setminus X$ [4].

PROPOSITION 2.4. *Let $\mathscr{A}(\tau)$ have the structure of a topological semilattice, and let a be a maximal idempotent of $\mathscr{A}(\tau)$. Then $\tau \leq \omega$.*

PROOF. Case 1. Suppose a is an identity of the semilattice $\mathscr{A}(\tau)$ and $\tau > \omega$. For any $a \in \mathscr{A}(\tau) \setminus \{a\}$, the set $\uparrow e$ is open in $\mathscr{A}(\tau)$ and hence, the set $\mathscr{A}(\tau) \setminus \uparrow e$ is finite. Thus the set $\mathscr{A}(\tau) \setminus \{a\}$ contains a countable chain $e_1 < e_2 < \cdots < e_n < \cdots$. Since for every $i \in \mathbb{N}$ the set $\mathscr{A}(\tau) \setminus \uparrow e_i$ is finite, then $|\bigcup_{i \in \mathbb{N}}(\mathscr{A}(\tau) \setminus \uparrow e_i)| \leq \omega$. Therefore, there exists an idempotent $e^* \in \mathscr{A}(\tau) \setminus \{a\}$ such that $e_i < e^*$ for any $i \in \mathbb{N}$, a contradiction with the inequality $|{\uparrow}e^*| < \omega$. If $\tau > \omega$, then a is not an identity of $\mathscr{A}(\tau)$.

Case 2. Suppose a is a maximal idempotent of $\mathscr{A}(\tau)$.

First, we shall prove that if $\tau > \omega$, then the set $\mathrm{Max}(\mathscr{A}(\tau))$ is infinite. Assume the contrary. Then a is an identity of the semilattice $\mathscr{A}(\tau) \setminus {\downarrow}(\mathrm{Max}(\mathscr{A}(\tau) \setminus \{a\}))$. Since the space ${\downarrow}(\mathrm{Max}(\mathscr{A}(\tau) \setminus \{a\}))$ is compact the space $\mathscr{A}(\tau) \setminus {\downarrow}(\mathrm{Max}(\mathscr{A}(\tau) \setminus \{a\}))$ is homeomorphic to the one-point Alexandroff compactification of uncountable discrete space. A contradiction with Case 1.

Further, we shall prove that if $\tau > \omega$ then $\mathscr{A}(\tau) \setminus \{a\} = {\downarrow}(\mathrm{Max}(\mathscr{A}(\tau) \setminus \{a\}))$. The inclusion ${\downarrow}(\mathrm{Max}(\mathscr{A}(\tau) \setminus \{a\})) \subseteq \mathscr{A}(\tau) \setminus \{a\}$ is trivial. Suppose that $\mathscr{A}(\tau) \setminus \{a\} \nsubseteq {\downarrow}(\mathrm{Max}(\mathscr{A}(\tau) \setminus \{a\}))$. Then there exists $e \in {\downarrow}a \setminus \{a\}$ such that $e \notin {\downarrow}(\mathrm{Max}(\mathscr{A}(\tau) \setminus \{a\}))$. Since $a \in {\uparrow}e$ and ${\uparrow}e$ is an open subset in $\mathscr{A}(\tau)$, then $|\mathscr{A}(\tau) \setminus {\uparrow}e| < \omega$. A contradiction with $|\mathrm{Max}(\mathscr{A}(\tau))| \geq \omega$. Thus the equality $\mathscr{A}(\tau) \setminus \{a\} = {\downarrow}(\mathrm{Max}(\mathscr{A}(\tau) \setminus \{a\}))$ holds.

We shall prove that $|{\downarrow}a| < \omega$. Suppose not. Then for any infinite chain $e_1 < e_2 < \cdots < e_n < \cdots < a$ there exists $e \in {\downarrow}a$ such that $e_i < e$ for any $i \in \mathbb{N}$. Hence $\{e_i \mid i \in \mathbb{N}\} \subseteq {\downarrow}e$, but $|{\downarrow}e| < \omega$. A contradiction. Thus $|{\downarrow}a| \leq \omega$.

Obviously, there exists a countable chain $e_1 < e_2 < \cdots < e_n < \cdots$ such that $e_i < a$ for any $i \in \mathbb{N}$. Since $|\mathscr{A}(\tau) \setminus {\uparrow}e_i| < \omega$ for every $i \in \mathbb{N}$ and $\mathscr{A}(\tau) \setminus \{a\} = {\downarrow}(\mathrm{Max}(\mathscr{A}(\tau) \setminus \{a\}))$, then $|\mathrm{Max}(\mathscr{A}(\tau))| \leq \omega$. For any $e \in \mathrm{Max}(\mathscr{A}(\tau) \setminus \{a\})$ the set ${\downarrow}e$ is finite, hence $|\mathrm{Max}(\mathscr{A}(\tau))| = |\mathscr{A}(\tau)| = |\mathscr{A}(\tau) \setminus \{a\}| = \tau$. So, if a is a maximal idempotent of semilattice $\mathscr{A}(\tau)$, then $\tau = \omega$. $\square$

Proposition 2.5. *Suppose that on $\mathscr{A}(\tau)$ $(\tau \geq \omega)$ there exists a structure of a topological semilattice. Then the following conclusions hold*:

(i) $|NO_{\mathscr{A}(\tau)}(a)| \leq \omega$.
(ii) *If $x \in NO_{\mathscr{A}(\tau)}(a)$, then the set ${\downarrow}x$ is finite.*
(iii) *If $x \in \mathrm{Max}(\mathscr{A}(\tau))$ and $a < x$, then a maximal chain $a < \cdots < x$ in $\mathscr{A}(\tau)$ is countable.*

Proof. (i) We define $A = (\mathscr{A}(\tau) \setminus {\uparrow}a) \bigcup \{a\}$. Then the topological subspace $A \subseteq \mathscr{A}(\tau)$ is homeomorphic to the one-point Alexandroff compactification of the discrete space of cardinality $\tau' \leq \tau$, and A is a compact topological semilattice such that a is a maximal idempotent of the semilattice A. By Proposition 2.4 we get $\tau' \leq \omega$ and, hence, $|NO_{\mathscr{A}(\tau)}(a)| \leq \omega$.

The proof of statement (ii) is trivial.

(iii) Suppose to the contrary, that there exists an uncountable chain $a < \cdots < x$. Since for any $g \in {\uparrow}a \setminus \{a\}$ the set ${\uparrow}g$ is finite, then there exists x_1 such that $a < x_1 < x$. Further, by induction for every integer $i \geq 2$ choose an indempotent x_i such that $a < x_i < x_{i-1} < x$. Put $\mathscr{M}(a) = \bigcup_{i \in \mathbb{N}} {\uparrow}x_i$. Since the chain $a < \cdots < x$ is uncountable, then there exists $y \in {\uparrow}a \cap {\downarrow}x$ such that $y \notin \mathscr{M}(a)$ and $a < y < x_i$ for any $i \in \mathbb{N}$. But the set $\uparrow y$ is finite, a contradiction. □

There exists no structure of a topological inverse semigroup with open left (right) principal ideals on the one-point Alexandroff compactification of an uncountable discrete space [6, Proposition 4.10].

The following example shows that there exists a topological semilattice on $\mathscr{A}(\tau)$ which satisfies statements of Proposition 2.4 and Proposition 2.5.

EXAMPLE 5. Let X be a discrete space of cardinality $\tau \geq \omega$, $\mathscr{N}$ the discrete space of natural numbers, and $\{0, 1\}$ a two-point discrete space. Further, we suppose that $\mathscr{A}(\tau)\setminus\{a\} = (X \times \mathscr{N})\bigcup(\{0, 1\} \times \mathscr{N})$. On $\mathscr{A}(\tau)$ we define the semilattice operation '$\star$' as follows:

(a) $x \star x = x$ for any $x \in \mathscr{A}(\tau)$.

(b) If $x, y \in X \times \mathscr{N}$ and $x = (x^o, m)$, $y = (y^o, n)$, then

$$x \star y = y \star x = \begin{cases} (x^o, \max\{m, n\}) & \text{if } x^o = y^o; \\ a & \text{if } x^o \neq y^o. \end{cases}$$

(c) If $x \in X \times \mathscr{N}$, then $x \star a = a \star x = a$.

(d) If $x, y \in \{0, 1\} \times \mathscr{N}$ and $x = (x', m)$, $y = (y', n)$, then

$$x \star y = y \star x = \begin{cases} x & \text{if } x = y; \\ (0, \min\{m, n\}) & \text{if } x \neq y. \end{cases}$$

(e) If $x \in \{0, 1\} \times \mathscr{N}$ and $x = (x^1, n)$, then $x \star a = a \star x = (0, n) \in \{0, 1\} \times \mathscr{N}$.

(f) If $x = (x^1, n) \in \{0, 1\} \times \mathscr{N}$ and $y = (y^1, m) \in X \times \mathscr{N}$, then $x \star y = y \star x = (0, n) \in \{0, 1\} \times \mathscr{N}$.

Obviously, $(\mathscr{A}(\tau), \star)$ is a topological semilattice.

Proposition 2.4 and Proposition 2.5 imply

THEOREM 2.6. *There exists no structure with a topological lattice on the one-point Alexandroff compactification of an uncountable discrete space.*

Item (i) of Proposition 2.5 implies

COROLLARY 2.7. *Let there exist on $\mathscr{A}(\tau)$ $(\tau \geq \omega)$ the structure of topological semilattice with open principal filters, then the set $NO_{\mathscr{A}(\tau)}(a)$ is finite.*

Example 6 shows that there exists a topological semilattice structure with open principal filters on $\mathscr{A}(\tau)$ which satisfies statements of Propositions 2.4–2.5 and Corollary 2.7.

EXAMPLE 6. Let X, $\mathscr{N}$ and $\{0,1\}$ be as in Example 5, and $L=\{1,2,\ldots,n\}$ be a discrete space. Further, we suppose that $\mathscr{A}(\tau)\setminus\{a\}=(X\times\mathscr{N})\bigcup(\{0,1\}\times L)$. On $\mathscr{A}(\tau)$ we define the semilattice operation '$\circ$' as follows:

(a) $x\circ x=x$ for any $x\in\mathscr{A}(\tau)$.

(b) If $x,y\in X\times\mathscr{N}$ and $x=(x^o,m)$, $y=(y^o,n)$, then

$$x\circ y=y\circ x=\begin{cases}(x^o,\max\{m,n\}) & \text{if } x^o=y^o;\\ a & \text{if } x^o\neq y^o.\end{cases}$$

(c) If $x\in X\times\mathscr{N}$, then $x\circ a=a\circ x=a$.

(d) If $x,y\in\{0,1\}\times L$ and $x=(x',i)$, $y=(y',j)$, then

$$x\circ y=y\circ x=\begin{cases}x & \text{if } x=y;\\ (0,\min\{i,j\}) & \text{if } x\neq y.\end{cases}$$

(e) If $x\in\{0,1\}\times L$ and $x=(x',i)$, then $x\circ a=a\circ x=(0,i)\in\{0,1\}\times L$.

(f) If $x=(x^1,n)\in\{0,1\}\times L$ and $y=(y^1,m)\in X\times\mathscr{N}$, then $x\circ y=y\circ x=(0,n)\in\{0,1\}\times L$.

Obviously, $(\mathscr{A}(\tau),\circ)$ is a topological semilattice with open principal filters.

REMARK. Questions about the structure of topological semigroups on one-point compactifications were considered in [15, 18] and in other papers.

3. Topological inverse *bopf*-semigroups

Let S be an algebraic semigroup. For any $a\in S$ we denote

$$\mathscr{L}_d(a)=\{x\in S\mid \text{ there exists } y\in S^1 \text{ such that } xy=a\};$$
$$\mathscr{R}_d(a)=\{x\in S\mid \text{ there exists } y\in S^1 \text{ such that } yx=a\};$$
$$\mathscr{J}_d(a)=\{x\in S\mid \text{ there exist } y,z\in S^1 \text{ such that } yxz=a\}.$$

LEMMA 3.1. *Let a be a regular element of the semigroup S, then*

(i) $\mathscr{L}_d(a)=\{x\in S\mid$ *there exists $y\in S$ such that $xy=a$*$\}$,

(ii) $\mathscr{R}_d(a)=\{x\in S\mid$ *there exists $y\in S$ such that $yx=a$*$\}$.

PROOF. Suppose a is a regular element in S. Then there exists $z\in S$ such that $a=aza$. We put $a_1=az$ and $a_2=za$. Hence, $a=a_1a$ and $a=aa_2$. □

LEMMA 3.2. *An element a of the semigroup S is regular if and only if $\mathscr{L}_d(a)=\mathscr{L}_d(e)$ $[\mathscr{R}_d(a)=\mathscr{R}_d(e)]$ for some idempotent $e\in S$.*

PROOF. If a is a regular element of S, then $a = axa$ for some $x \in S$. Hence, $e = ax$ and $f = xa$ are idempotens of S such that $ea = a = af$. If $z \in \mathscr{L}_d(a)$ [$z \in \mathscr{R}_d(a)$], then, by Lemma 3.1, $a = zy$ [$a = yz$] for some $y \in S$. Hence, $e = ax = zax$ [$f = xa = xyz$] and $z \in \mathscr{L}_d(e)$ [$z \in \mathscr{R}_d(f)$]. If $w \in \mathscr{L}_d(e)$ [$w \in \mathscr{R}_d(f)$], then $e = wk$ [$f = kw$] for some $k \in S$. Thus, $a = ea = wka$ [$a = af = akw$], and, therefore, $w \in \mathscr{L}_d(a)$ [$w \in \mathscr{R}_d(a)$].

Suppose $\mathscr{L}_d(a) = \mathscr{L}_d(e)$ [$\mathscr{R}_d(a) = \mathscr{R}_d(e)$]. Then there exist $x, y \in S^1$ such that $a = ex$ and $e = ay$ [$a = xe$ and $e = ya$]. Hence, $ea = eex = ex = a$ [$ae = xee = xe = a$] and $a = ea = aya$ [$a = ae = aya$]. If y is an identity of S then $a = e$ and $a = ae = aea = aaa$ [$a = ea = eea = aaa$]. Thus $a \in aSa$. □

LEMMA 3.3. *A semigroup S is inverse if and only if the following conditions hold*:

(i) *For any $a \in S$ there exists an unique idempotent $e \in S$ such that $\mathscr{L}_d(a) = \mathscr{L}_d(e)$.*

(ii) *For any $a \in S$ there exists an unique idempotent $e \in S$ such that $\mathscr{R}_d(a) = \mathscr{R}_d(e)$.*

PROOF. Suppose that for some idempotents $e, f \in S$ the equality $\mathscr{R}_d(e) = \mathscr{R}_d(f)$ holds. Then there exist $x, y \in S$ such that $e = fx$ and $f = ey$. Since,

$$e = ee = ee^{-1} = fx(fx)^{-1} = fxx^{-1}f = ffxx^{-1} = fxx^{-1}$$

and

$$f = ff = ff^{-1} = ey(ey)^{-1} = eyy^{-1}e = eeyy^{-1} = eyy^{-1}$$

we have $e \leq f$ and $f \leq e$. Hence $e = f$.

Suppose the statements (i) and (ii) hold. By Lemma 3.2, the semigroup S is regular. Let $a \in S$, and suppose there exist distinct $b, c \in S$ such that

$$aba = a, \; bab = b, \; aca = a, \; cac = c.$$

Since $\mathscr{L}_d(a) = \mathscr{L}_d(ab) = \mathscr{L}_d(ac)$ and $\mathscr{R}_d(a) = \mathscr{R}_d(ba) = \mathscr{R}_d(ca)$ we have that $ba = ca$ and $ab = ac$. Hence, $b = bab = cab = cac = c$, and S is inverse semigroup. □

DEFINITION. A topological inverse semigroup S is called a *bopf*-semigroup if the band of S is a semilattice with open principal filters.

THEOREM 3.4. *Let S be a topological inverse semigroup. Then the following conditions are equivalent*:

(i) *S is a bopf-semigroup.*

(ii) *For every $a \in S$, the set $\mathscr{L}_d(a)$ is open in S.*

(iii) *For every $a \in S$, the set $\mathscr{R}_d(a)$ is open in S.*

PROOF. Implications (ii) implies (i) and (iii) implies (i) are trivial.

(i) implies (ii). We shall prove that for every $x \in S$ the equality $\varphi^{-1}(\uparrow(xx^{-1})) = \mathscr{L}_d(x)$ holds. Let $y \in \varphi^{-1}(\uparrow(xx^{-1}))$, then $yy^{-1} \in \uparrow(xx^{-1})$. Thus, $yy^{-1}xx^{-1} = xx^{-1}$ and $yy^{-1}xx^{-1}x = yy^{-1}x = x$. Hence, $y \in \mathscr{L}_d(x)$. Therefore, we get $\varphi^{-1}(\uparrow(xx^{-1})) \subseteq \mathscr{L}_d(x)$.

Let $y \in \mathscr{L}_d(x)$ then there exists $b \in S$ such that $x = yb$. Hence, $xx^{-1}ybb^{-1}y^{-1}$ and $xx^{-1} = xx^{-1}xx^{-1} = ybb^{-1}y^{-1}xx^{-1}$. Thus $ybb^{-1}y^{-1} \in \uparrow(xx^{-1})$. Since $yy^{-1} \in \uparrow(ybb^{-1}y^{-1})$, then $yy^{-1} \in \uparrow(xx^{-1})$ and hence $yy^{-1} \in \varphi^{-1}(\uparrow(xx^{-1}))$. Therefore, $\mathscr{L}_d(x) \subseteq \varphi^{-1}(\uparrow(xx^{-1}))$.

The implication (i) implies (iii) follows from $\psi^{-1}(\uparrow(xx^{-1})) = \mathscr{R}_d(x)$. □

THEOREM 3.5. *Let S be a topological inverse Clifford semigroup. Then S is a bopf-semigroup if and only if the set $\mathscr{J}_d(a)$ is open in S for every $a \in S$.*

PROOF. If for every $a \in S$, the set $\mathscr{J}_d(a)$ is open in S, then the band of S is a semilattice with open principal filters.

Suppose S is a *bopf*-semigroup and E is a band of S. Since S is a Clifford inverse semigroup, the maps $\varphi\colon S \to E$ and $\psi\colon S \to E$ coincide. We shall prove that $\mathscr{J}_d(a) = \varphi^{-1}(\uparrow(aa^{-1}))$ for every $a \in S$. If $x \in \mathscr{J}_d(a)$, then there exist $y, z \in S$ such that $yxz = a$. By [16, Theorem II.26], we have

$$aa^{-1} = yxz(yxz)^{-1} = yxzz^{-1}x^{-1}y^{-1} = zz^{-1}yxx^{-1}y^{-1} = zz^{-1}yy^{-1}xx^{-1}.$$

Hence, $xx^{-1} \in \uparrow(aa^{-1})$. Therefore, $\mathscr{J}_d(a) \subseteq \varphi^{-1}(\uparrow(aa^{-1}))$.

If $x \in \varphi^{-1}(\uparrow(aa^{-1}))$, then $xx^{-1} \in \uparrow(aa^{-1})$ and there exists $e \in E$ such that $exx^{-1} = aa^{-1}$, that is, $exx^{-1}a = a$; hence, $x \in \mathscr{J}_d(a)$. Therefore, $\varphi^{-1}(\uparrow(aa^{-1})) \subseteq \mathscr{J}_d(a)$. □

The following example shows that there exists a topological inverse semigroup S such that the set $\mathscr{J}_d(s)$ is open in S for every $s \in S$ and S is not a *bopf*-semigroup.

EXAMPLE 7. Let S be an inverse semigroup and $a, b \notin S$. A semigroup $\mathscr{C}(S)$ is generated by the set $S \cup \{a, b\}$ and is defined by the following equalities: $ab = 1$, $as = a$, $sb = b$ and by equalities in S. If S has the identity, then the identity of $\mathscr{C}(S)$ is the identity of S. In the other case the identity of $\mathscr{C}(S)$ is an accessory identity of S (see [3, Section 1.1]). Any element of $\mathscr{C}(S)$ is uniquely represented by $b^i t a^j$, $t \in S \cup \{1\}$, $i, j \in \mathbb{N} \cup \{0\}$.

Let S be a topological inverse semigroup. If S has no identity, let $S^1 = S \cup \{1\}$ be a semigroup with an isolated accessory identity. Let $\mathscr{B}$ be a base of the topology on S^1. A topology τ on $\mathscr{C}(S)$ is determined by the base

$$\mathscr{B}_{\mathscr{C}} = \{b^i U a^j \mid U \in \mathscr{B}, i, j \in \mathbb{N} \cup \{0\}\}.$$

By [7, Corollary 1] $(\mathscr{C}(S), \tau)$ is a simple topological inverse semigroup and S is topologically isomorphically embedded into $(\mathscr{C}(S), \tau)$. The semigroup $(\mathscr{C}(S), \tau)$ is called the *Bruck semigroup over* S [7].

Let S be a topological inverse semigroup which is not a *bopf*-semigroup. Let $\mathscr{C}(S)$ to be the Bruck semigroup over S. Obviously, $\mathscr{J}_d(s) = \mathscr{C}(S)$ for any $s \in \mathscr{C}(S)$ and, hence, the set $\mathscr{J}_d(s)$ is open in $\mathscr{C}(S)$ for all $s \in S$. However, the band of $\mathscr{C}(S)$ is not a semilattice with open principal filters.

THEOREM 3.6. *Every first countable compact inverse bopf-semigroup is metrizable.*

PROOF. Let S be as in the statement. The band $E(S)$ is a first countable space. Let $e, f \in E(S)$. If $H(e, f) \neq \varnothing$, then $H(e, f)$ is homeomorphic to the metrizable subgroup $H(e)$ and, hence, $H(e, f)$ is a metrizable compactum. Theorem 1.2 implies $|E(S)| = \chi(E(S)) \leq \omega$, hence, S is a countable union of metrizable compacta and by the Arhangelskii Theorem (see [4, Theorem 3.2.20]) S is metrizable. □

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