Journal of Mathematical Sciences, Vol. 104, No. 5, 2001

EMBEDDING OF COUNTABLE TOPOLOGICAL SEMIGROUPS IN SIMPLE COUNTABLE CONNECTED TOPOLOGICAL SEMIGROUPS

O. V. Gutik UDC 512.536.72

We prove that any countable Hausdorff topological (inverse) semigroup is topologically isomorphically embedded into a simple countable connected Hausdorff topological (inverse) semigroup with identity.

This work is a sequel to the author's investigations in [1] and [2].

The terminology and notation are the same as in [3, 4, 8]. All topological spaces are regarded as Hausdorff. The sets of natural, integer, and nonnegative integer numbers are denoted, respectively, by $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Z}_+$, and $\mathfrak{c}$ denotes the cardinal number of continuum.

An arbitrary topological (inverse) semigroup is topologically isomorphically embedded in a simple linearly connected topological (inverse) semigroup [2]. A natural question arises about the analog of this result for countable semigroups: *is it true that an arbitrary countable Hausdorff topological semigroup is embedded in a simple countable connected Hausdorff topological semigroup?* We answer this question in the affirmative. The construction proposed is connected with the use of the Bruck algebraic construction for embedding an arbitrary semigroup in a simple semigroup with identity [6]. Let S be a semigroup and let $a, b \notin S$. A semigroup $C(S)$ is generated by the set $S \bigcup \{a, b\}$ and is defined by the relations $ab = 1$, $as = a$, and $sb = b$ for all $s \in S$ and also by the relations valid in S. The identity 1 of the semigroup $C(S)$ is either the identity of the semigroup S if $1 \in S$ or the identity added to $C(S)$ in the usual way if S does not contain the identity. The semigroup $C(S)$ constructed in such a way will be called the Bruck semigroup over S [1]. Every element of the semigroup $C(S)$ is uniquely represented in the form $b^i t a^j$, where $t \in S^1$, $i, j \in \mathbb{Z}_+$ [6]. The semigroup S is algebraically embedded in $C(S)$, and $C(S)$ is a simple semigroup [6].

Let X be a countable connected Hausdorff space (see, e.g., [5] or [11]). Consider the Lawson free semilattice $\exp_\omega(X)$ over X [9]. In what follows, we assume that all elements of the semilattice $\exp_\omega(X)$ are uncontractible words. Let $\mathcal{I}_n(X)$, $n \in \mathbb{N}$, be a subspace of the topological space $\exp_\omega(X)$ formed by all nonempty subsets in $\exp_\omega(X)$ of power $\leq n$. Since $\mathcal{I}_n(X)$ is a factor space X^n for arbitrary $n \in \mathbb{N}$ [10] and since $\exp_\omega(X) = \bigcup_{n=1}^{\infty} \mathcal{I}_n(X)$, we have

Proposition 1. *A Lawson free semilattice over a countable connected Hausdorff space is a Hausdorff countable connected topological semilattice.*

We fix an arbitrary element x_0 in X and denote

$$\exp_\omega\{x_0\}(X) = \{ T \in \exp_\omega(X) \mid x_0 \in T \}.$$

Proposition 2. *Let X be a countable connected Hausdorff topological space. Then $\exp_\omega\{x_0\}(X)$ is a countable connected topological semilattice in $\exp_\omega(X)$.*

Proof. It is obvious that $\exp_\omega\{x_0\}(X)$ is a topological subsemilattice in $\exp_\omega(X)$. We define a mapping $\varphi\colon \exp_\omega(X) \to \exp_\omega(X)$ as follows: $\varphi(T) = T \bigcup \{x_0\}$ for arbitrary $T \in \exp_\omega\{x_0\}(X)$. Since φ is a continuous

Pidstryhach Institute of Applied Problems in Mechanics and Mathematics, Ukrainian Academy of Sciences, L'viv. Translated from Matematychni Metody ta Fizyko-Mekhanichni Polya, Vol. 41, No. 3, pp. 16–21, July–September, 1998. Original article submitted July 13, 1998.

 1072–3374/01/1045–1422 $25.00

retraction and $\varphi(\exp_\omega(X)) = \exp_\omega\{x_0\}(X)$, it follows from Proposition 1 that $\exp_\omega\{x_0\}(X)$ is a closed countable connected topological subsemilattice in $\exp_\omega(X)$.

Let S be an arbitrary countable Hausdorff topological (inverse) semigroup. If S does not contain the identity, then we assume that the identity is discretely added to S. Denote

$$G = S \times \exp_\omega\{x_0\}(X)$$

with the topology of a Cartesian product. In what follows, by $\mathcal{B}_G(s)$, we denote the base of a topology at the point $s \in G$.

The proof of the following statement is trivial:

Proposition 3. *If S is a countable Hausdorff topological (inverse) semigroup, then G is also a countable Hausdorff topological (inverse) semigroup.*

Note that $1_G = (1_S, \{x_0\})$ is the identity of the semigroup G, where 1_S is the identity of the semigroup S.

For arbitrary $x \in \exp_\omega\{x_0\}(X)$, by $d(x)$ we denote the length of a word x. Denote $E_n = \{ x \in \exp_\omega\{x_0\}(X) \mid d(x) \geq n \}$ and $G_n = E_n \times S$ for every $n \in \mathbb{N}$.

Let $CG = C(G)$ be a Bruck semigroup over G and let τ_0^* be the topology of a direct sum on CG (see Theorem 1 of [1]). We weaken the topology τ_0^* to a semigroup topology $\tilde{\tau}_0$ as follows: The bases of the topologies $\tilde{\tau}_0$ and τ_0^* coincide at the points 1_G, b^i, a^j, $b^i s$, sa^j, and $b^i s a^j$, where $i, j \in \mathbb{N}$ and $s \in G \setminus \{1_G\}$. At the points $b^i a^j \in CG$, $i, j \in \mathbb{N}$, the base $\mathcal{B}(b^i a^j)$ of the topology $\tilde{\tau}_0$ is defined as follows. Let $\mathcal{B}_G(1_G)$ be the base of a topology at 1_G. Then the family

$$\tilde{\mathcal{B}}_0(b^i a^j) = \left\{ M_{i,j}(U, n) = b^{i-1} G_n a^{j-1} \cup b^i U a^j \mid U \in \mathcal{B}_G(1_G),\ n \in \mathbb{N} \right\}$$

satisfies conditions (BP1)–(BP3) of [4] and, hence, defines the base of the topology $\tilde{\tau}_0$ at the points $b^i a^j \in CG$, where $i, j \in \mathbb{N}$.

Lemma 1. *If S is a topological (inverse) semigroup, then $(CG, \tilde{\tau}_0)$ is also a topological (inverse) semigroup.*

Proof. Since the direct-sum topology τ_0^* on CG is a semigroup topology (see Theorem 1 in [1]), it suffices, obviously, to check the continuity of multiplication in (CG, τ_0^*) in the following three cases:

1) $b^i s a^j \cdot b^k a^l$,

2) $b^k a^l \cdot b^i s a^j$,

3) $b^k a^l \cdot b^m a^n$,

where $i, j \in \mathbb{Z}_+$; $k, l, m, n \in \mathbb{N}$; $s \in G$ if $ij = 0$, and $s \in G \setminus \{1_G\}$ if $i, j \in \mathbb{N}$.

We consider all these cases.

$$1) \qquad b^i s a^j \cdot b^k a^l = \begin{cases} b^i s a^{j-k+l} & \text{if } j \geq k, \\ b^{i+k-j} s a^l & \text{if } j < k. \end{cases}$$

Hence,

(a) if $j \geq k$, then $b^i U(s) a^j \cdot M_{k,l}(V, n) \subseteq b^i U(s) a^{j-k+l}$ for arbitrary $U(s) \in \mathcal{B}_G(s)$, $V \in \mathcal{B}_G(1_G)$, and $n \in \mathbb{N}$;

(b) if $j < k$, then $b^i W(s) a^j \cdot M_{k,l}(U, n) \subseteq M_{i+k-j,\, l}(U, n)$ for arbitrary $W(s) \in \mathcal{B}_G(s)$, $U \in \mathcal{B}_G(1_G)$, and $n \in \mathbb{N}$.

2) $$b^k a^l \cdot b^i s a^j = \begin{cases} b^{i-l+k} s a^j, & \text{if } l \leq i, \\ b^k a^{l-i+j}, & \text{if } l > i. \end{cases}$$

Therefore,

(a) if $l \leq i$, then $M_{k,l}(V, n) \cdot b^i U(s) a^j \subseteq b^{i-l+k} U(s) a^j$ for arbitrary $U(s) \in \mathcal{B}_G(s)$, $V \in \mathcal{B}_G(1_G)$, and $n \in \mathbb{N}$;

(b) if $l > i$, then $M_{k,l}(U, n) \cdot b^i W(s) a^j \subseteq M_{k,\, l-i+j}(U, n)$ for arbitrary $W(s) \in \mathcal{B}_G(s)$, $U \in \mathcal{B}_G(1_G)$, and $n \in \mathbb{N}$.

3) $$b^k a^l \cdot b^m a^n = \begin{cases} b^{k-l+m} a^n, & \text{if } l \leq m, \\ b^k a^{l-m+n}, & \text{if } l > m. \end{cases}$$

Let the condition $U(1_G) \cdot U(1_G) \subseteq W(1_G)$ be satisfied for the neighborhoods $U(1_G)$ and $W(1_G)$ in G; then, for arbitrary $p \in \mathbb{N}$, we have

(a) if $l \leq m$, then $M_{k,l}(U(1_G), p) \cdot M_{m,n}(U(1_G), p) \subseteq M_{k-l+m,\, n}(W(1_G), 2p)$;

(b) if $l > m$, then $M_{k,l}(U(1_G), p) \cdot M_{m,n}(U(1_G), p) \subseteq M_{k,\, l-m+n}(W(1_G), 2p)$.

Let S be a topological inverse semigroup. Since the direct-sum topology τ_0^* on CG is a semigroup inverse topology (see Corollary 1 in [1]), it suffices to check the continuity of the inversion in $(CG, \tilde{\tau}_0)$ only at the points $b^i a^j$, where $i, j \in \mathbb{N}$. If $(U(1_G))^{-1} \subseteq V(1_G)$ in G, then $\left(M_{i,j}(U(1_G), p)\right)^{-1} \subseteq M_{j,i}(V(1_G), p)$ for arbitrary $p \in \mathbb{N}$.

Next, we perform a reasoning similar to that in [2]. Let $C^0(S) = CG$. By setting $C^n(S) = C\left(C^{n-1}(S)\right)$, we obtain a family of semigroups $\left\{C^n(S) \mid n \in \mathbb{N}\right\}$. In what follows, by 1_n, where $n \in \mathbb{Z}_+$, we denote the identity of the semigroup $C^n(S)$.

On the semigroup $C^1(S)$, the direct-sum topology τ_1^* (see Theorem 1 in [1]) will be weakened to a semigroup topology $\tilde{\tau}_1$ as follows. At the points 1_0, $b_1^{i_1}$, $a_1^{j_1}$, s, $b_1^{i_1} s$, $s a_1^{j_1}$, and $b_1^{i_1} s a_1^{j_1}$, where $i_1, j_1 \in \mathbb{N}$ and $s \in C^0(S) \setminus \{1_0\}$, the bases of the topologies τ_1^* and $\tilde{\tau}_1$ coincide. We weaken the topology τ_1^* only at the points $b_1^{i_1} 1_0 a_1^{j_1}$, where $i_1, j_1 \in \mathbb{N}$. For arbitrary $m \in \mathbb{N}$, we denote

$$S_0(m) = \left\{ b^k g a^l \,\middle|\, k \ge m,\ l \ge m,\ g \in G \right\}$$

$$\setminus \left(\bigcup \left\{ b^m(\{s\} \times \{x_0\}) a^i \,\middle|\, i \ge m,\ s \in S \right\} \bigcup \bigcup \left\{ b^i(\{s\} \times \{x_0\}) a^m \,\middle|\, i \ge m,\ s \in S \right\} \right).$$

Let $\tilde{\mathcal{B}}_0(1_0)$ be the base of the topology $\tilde{\tau}_0$ at the point $1_0 \in C^0(S)$; then the family

$$\tilde{\mathcal{B}}_1\left(b_1^{i_1} a_1^{j_1}\right) = \left\{ M_1^{i_1, j_1}(U, m) = b_1^{i_1} U(1_0) a_1^{j_1} \cup b_1^{i_1 - 1} S_0(m) a_1^{j_1 - 1} \,\middle|\, U(1_0) \in \tilde{\mathcal{B}}_0(1_0),\ m \in \mathbb{N} \right\}$$

satisfies conditions (BP1)–(BP3) in [4] and, hence, defines the base of the topology $\tilde{\tau}_1$ at the points $b_1^{i_1} a_1^{j_1}$, where $i_1,\ j_1 \in \mathbb{N}$.

Assume that the topology $\tilde{\tau}_{n-1}$ is already defined on the semigroup $C^{n-1}(S)$. Now we define the topology $\tilde{\tau}_n$ on $C^n(S)$. The direct-sum topology τ_n^* on $C^n(S)$ (see Theorem 1 in [1]) will be weakened to a semigroup topology $\tilde{\tau}_n$ as follows. At the points 1_{n-1}, $b_n^{i_n}$, $a_n^{j_n}$, s, $b_n^{i_n} s$, $s a_n^{j_n}$, and $b_n^{i_n} s a_n^{j_n}$, where $i_n,\ j_n \in \mathbb{N}$ and $s \in C^{n-1}(S) \setminus \{1_{n-1}\}$, the bases of the topologies τ_n^* and $\tilde{\tau}_n$ coincide. We weaken the topology τ_n^* only at the points $b_n^{i_n} a_n^{j_n}$, where $i_n,\ j_n \in \mathbb{N}$. For arbitrary $m \in \mathbb{N}$, we denote

$$S_{n-1}(m) = \left\{ b_{n-1}^k g a_{n-1}^l \,\middle|\, k \ge m,\ l \ge m,\ g \in C^{n-2}(S) \right\}$$

$$\setminus \left(\bigcup \left\{ b_{n-1}^m b_{n-2}^{i_{n-2}} \ldots b_1^{i_1} b^{i_0} (\{s\} \times \{x_0\}) a^{j_0} a_1^{j_1} \ldots a_{n-2}^{j_{n-2}} a_{n-1}^k \,\middle|\, i_p \cdot j_p = 0,\ p = 0, \ldots, n-2,\ k \ge m,\ s \in S \right\} \right.$$

$$\left. \bigcup \bigcup \left\{ b_{n-1}^k b_{n-2}^{i_{n-2}} \ldots b_1^{i_1} b^{i_0} (\{s\} \times \{x_0\}) a^{j_0} a_1^{j_1} \ldots a_{n-2}^{j_{n-2}} a_{n-1}^m \,\middle|\, i_p \cdot j_p = 0,\ p = 0, \ldots, n-2,\ k \ge m,\ s \in S \right\} \right).$$

Let $\tilde{\mathcal{B}}_{N-1}(1_{n-1})$ be the base of the topology $\tilde{\tau}_{n-1}$ at the point $1_{n-1} \in C^{n-1}(S)$. The family

$$\tilde{\mathcal{B}}_N\left(b_n^{i_n} a_n^{j_n}\right) = \left\{ M_n^{i_n, j_n}(U, m) = b_n^{i_n} U a_n^{j_n} \cup b_n^{i_n - 1} S_{n-1}(m) a_n^{j_n - 1} \,\middle|\, U \in \tilde{\mathcal{B}}_{N-1}(1_{n-1}),\ m \in \mathbb{N} \right\}$$

satisfies conditions (BP1)–(BP3) in [4] and, hence, defines the base of the topology $\tilde{\tau}_n$ at the points $b_n^{i_n} a_n^{j_n}$, where $i_n,\ j_n \in \mathbb{N}$.

Lemma 2. *If S is a topological (inverse) semigroup, then $\left\{ \left(C^n(S), \tilde{\tau}_n\right) \,\middle|\, n \in \mathbb{N} \right\}$ is a family of simple topological (inverse) semigroups.*

The proof of Lemma 2 coincides with the proof of Lemma 2 in [2].

For arbitrary $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, denote $L_k^0 = \left\{ b^i g a^j \,\middle|\, i - j = k,\ g \in G \right\}$ and $L_k^n = \left\{ b_n^{i_n} t a_n^{j_n} \mid i_n - j_n = k,\ t \in C^{n-1}(S) \right\}$.

Proposition 4. *For arbitrary $i \in \mathbb{Z}_+$, $k,\ p \in \mathbb{Z}$, the subspaces L_p^i and L_k^i in $\left(C^{i+1}(S), \tilde{\tau}_{i+1}\right)$ are homeomorphic.*

Proof. It obviously suffices to show the homeomorphicity of the spaces L_0^i and L_k^i.

Let $k > 0$. We denote the maps $a_k : C^{i+1}(S) \to C^{i+1}(S)$ and $b_k : C^{i+1}(S) \to C^{i+1}(S)$ as follows: $a_k(x) = a^k x$ and $b_k(x) = b^k x$. Then the map $f = a_k \big|_{L_k^i} : L_k^i \to L_0^i$ is a homeomorphism since $f^{-1} = b_k \big|_{L_0^i}$ and $\left(C^{i+1}(S), \tilde{\tau}_{i+1}\right)$ is a topological semigroup. If $k < 0$, then $a_k(x) = xb^k$ and $b_k(x) = xa^k$.

Lemma 3. *For any* $k \in \mathbb{Z}_+$,, *the semigroup* L_0^k *is a connected subsemigroup of the topological semigroup* $\left(C^{k+1}(S), \tilde{\tau}_{k+1}\right)$.

Proof. Let $k = 0$. It suffices to show that $S_0^i = \left\{ b^i s a^i \,\middle|\, s \in G \right\} \bigcup \left\{ b^{i+1} a^{i+1} \right\}$ is a connected subsemigroup in $\left(C^0(S), \tilde{\tau}_0\right)$. Since, for arbitrary $i \in \mathbb{N}$: $S_0^i \bigcap S_0^{i+1} = \left\{ b^{i+1} a^{i+1} \right\}$, S_0^i and S_0^0 are homeomorphic subspaces in $\left(C^0(S), \tilde{\tau}_0\right)$, and since $L_0^0 = \bigcup_{n=1}^{\infty} S_0^n$, the connectedness of S_0^0 implies the connectedness of L_0^0.

Suppose that S_0^0 is a disconnected subspace in $\left(C^0(S), \tilde{\tau}_0\right)$. Then there exist $A, B \in \tilde{\tau}_0 \big|_{S_0^0}$ such that $S_0^0 = A \bigcup B$, $A \bigcap B = \varnothing$, $A \neq \varnothing$, $B \neq \varnothing$. Let $ba \in B$. Then $B \bigcap (S \times \exp_\omega\{x_0\}(X)) \neq \varnothing$. For arbitrary $s \in S$, denote $A(s) = A \bigcap (S \times \exp_\omega\{x_0\}(X))$ and $B(s) = B \bigcap (S \times \exp_\omega\{x_0\}(X))$. Hence,

1) $A(s) \neq \varnothing$ and $B(s) \neq \varnothing$,

2) $A(s) \bigcap B(s) = \varnothing$,

3) $A(s) \bigcup B(s) = \{s\} \times \exp_\omega\{x_0\}(X)$,

4) $A(s)$ and $B(s)$ are open subsets in $\{s\} \times \exp_\omega\{x_0\}(X)$

for any $s \in S$, what contradicts Proposition 1 since the subspace $\{s\} \times \exp_\omega\{x_0\}(X)$ in $\left(C^0(S), \tilde{\tau}_0\right)$ is homeomorphic to $\exp_\omega\{x_0\}(X)$.

Suppose that the statement of Lemma 3 is satisfied for $k < n$. Since the connectedness of the subsemigroup $S_n^i = \left\{ b_n^i a_n^j \,\middle|\, s \in C^{n-1}(S) \right\} \bigcup \left\{ b_n^{i+1} a_n^{i+1} \right\}$, $i \in \mathbb{Z}_+$, implies the connectedness of the semigroup L_0^n and since S_n^0 and S_n^i are homeomorphic subspaces in $\left(C^n(S), \tilde{\tau}_n\right)$ for arbitrary $i \in \mathbb{N}$, it suffices to prove the connectedness of the semigroup S_n^0. Let S_n^0 be not a connected subsemigroup in $\left(C^n(S), \tilde{\tau}_n\right)$. Then there exist A_n, $B_n \in \tilde{\tau}_n$ such that $S_n^0 \subset A_n \bigcup B_n$, $A_n \bigcap B_n = \varnothing$, $A_n \bigcap L_0^{n-1} \neq \varnothing$, and $B_n \bigcap L_0^{n-1} \neq \varnothing$. Denote $\tilde{A}_n = A_n \bigcap L_0^{n-1}$ and $\tilde{B}_n = B_n \bigcap L_0^{n-1}$. Then

1) $\tilde{A}_n \neq \varnothing$ and $\tilde{B}_n \neq \varnothing$,

2) $\tilde{A}_n \bigcap \tilde{B}_n = \varnothing$,

3) $\tilde{A}_n \bigcup \tilde{B}_n = L_0^{n-1}$,

4) $\tilde{A}_n$ and $\tilde{B}_n$ are open subsets in L_0^{n-1},

which contradicts the assumption of connectedness of L_0^{n-1}. This contradiction implies the statement of Lemma 3.

The following theorem gives an affirmative answer to the question posed by I. V. Protasov, namely, *whether there exist countable connected topological inverse semigroups:*

Theorem 1. *An arbitrary countable topological (inverse) semigroup is topologically isomorphically embedded in a countable simple connected Hausdorff topological (inverse) semigroup with identity.*

Proof. The family $\left\{\left(\mathcal{C}^n(S), \tau_n^*\right) \,\middle|\, n \in \mathbb{Z}_+\right\}$ satisfies the conditions of Proposition 1 in [2]. We set $\mathcal{C}^{\omega}(S) = \bigcup\left\{\mathcal{C}^n(S) \,\middle|\, n \in \mathbb{Z}_+\right\}$. The topology τ on $\mathcal{C}^{\omega}(S)$ is generated by the family $\left\{\mathcal{B}^{\omega}(s) \,\middle|\, s \in \mathcal{C}^{\omega}(S)\right\}$, where

$$\mathcal{B}^{\omega}(s) = \bigcup_{n\in\mathbb{Z}_+}\left\{\tilde{\mathcal{B}}_n(s) \,\middle|\, \tilde{\mathcal{B}}_n(s) \text{ is the base of the topology } \tilde{\tau}_n \text{ at the point } s \in \bigcup_{i=1}^{\infty} \mathcal{C}^i(S)\right\}.$$

Since $\mathcal{C}^{\omega}(S) = \bigcup_{i=1}^{\infty} L_0^n$, it follows from Corollary 6.1.10 in [4] that $\left(\mathcal{C}^{\omega}(S), \tau\right)$ is a connected countable Hausdorff topological semigroup. In this case, if S is a topological inverse semigroup, then so is $\left(\mathcal{C}^{\omega}(S), \tau\right)$.

Corollary 1. *An arbitrary countable topological inverse Clifford semigroup is topologically isomorphically embedded in a countable connected Hausdorff topological inverse Clifford semigroup.*

Corollary 2. *An arbitrary countable topological semilattice is topologically isomorphically embedded in a countable connected Hausdorff topological semilattice.*

Note that all the results obtained for countable topological semigroups are directly applied to topological semigroups of power λ. However, the reasoning presented in [2] implies a stronger statement, namely, *an arbitrary topological (inverse) semigroup of power λ is topologically isomorphically embedded in a simple linearly connected topological (inverse) semigroup with identity of power* $\min\{\lambda, \mathfrak{c}\}$.

It is obvious that if one considers the Hartman–Mycielsky theorem [7] and its corresponding analog for simple topological semigroups [2], then there exists no analog of the results obtained in this paper in the category of topological groups since countable connected separated topological semigroups do not exist (see, e.g., Corollary 6.1.3 in monograph [4]).

REFERENCES

1. O. V. Gutik, "Embedding of topological semigroups in simple ones," *Mat. Studii,* No. 3, 10–14 (1994).
2. O. V. Gutik, "An arbitrary topological semigroup is topologically isomorphically embedded in a simple linearly connected topological semigroup," in: *Algebra and Topology* [in Ukrainian], L'viv University, L'viv (1996), pp. 65–73.
3. A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups,* American Mathematical Society, Providence (1964).
4. R. Engelking, *General Topology,* Pastwowe Wydawnictwo Naukowe, Warsaw (1977).
5. R. H. Bing, "A connected countable Hausdorff space," *Proc. Amer. Math. Soc.,* No. 4, 474 (1953).
6. R. H. Bruck, "A survey of binary systems," in: *Ergebnisse der Math.,* Heft 20, Springer, Berlin (1958), pp. 1–185.
7. S. Hartman and J. Mycielsky, "On the imbedding of topological groups into connected topological groups," *Colloq. Math.,* **5**, 167–169 (1958).
8. K. H. Hofmann and P. S. Mostert, *Elements of Compact Semigroups,* Columbus, Ohio (1966).
9. M. M. McWaters, "A note on topological semilattices," *J. London Math. Soc.,* **1**, No. 4, 64–69 (1969)
10. E. Michael, "Topologies on spaces of subsets," *Trans. Amer. Math. Soc.,* **71**, 152–182 (1951).
11. P. S. Urysohn, "Über die Mächtigkeit der zusammenhängenden Mengen," *Math. Ann.,* **94**, 262–295 (1925).